A note on Uniquely Pancyclic Graphs

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Abstract

In this paper we consider uniquely pancyclic graphs, ie n vertex graphs with exactly one cycle of each length from 3 to n.

The first result of the paper gives new upper and lower bounds on the number of edges in a uniquely pancyclic graph. Next we report on a computer search for new uniquely pancyclic graphs. We found that there are no new such graphs on $n \leq 59$ vertices and that there are no uniquely pancyclic graphs with exactly 5 chords.

1 Introduction

A graph G on n vertices is said to be pancyclic if it contains cycles of every length from 3 to n, [1], and is *uniquely pancyclic*, abbreviated UPC, if it contains exactly one cycle of every length from 3 to n. The question of for which n there exists a UPC graph on n vertices is usually claimed to have been asked by Entringer in 1973 and has since been well known as one of the 50 unsolved problems included at the end of Bondy and Murty's now classic textbook Graph Theory with Applications [2]. While several of these 50 open problems has been solved since this book first appeared it is not even known if there are infinitely many n for which a UPC graph on n vertices exist. In fact it has been conjectured that there are not, [6].

The simplest UPC graph is of course the complete graph on three vertices, K_3 and a bit of thought let you find the other three graphs in Fig.1 as well, these are the only UPC graphs with less than nine vertices. In [6] Shi proves that the four graphs in Fig.1 are the only outerplanar UPC graphs and finds three further UPC graphs, shown in Fig.2. Shi also proves that these seven graphs are the only UPC graphs obtainable from a hamiltonian cycle by adding two, three or four edges. The proof is based on a lengthy case analysis.

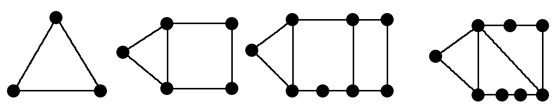

Figure 1: The first four UPC graphs.

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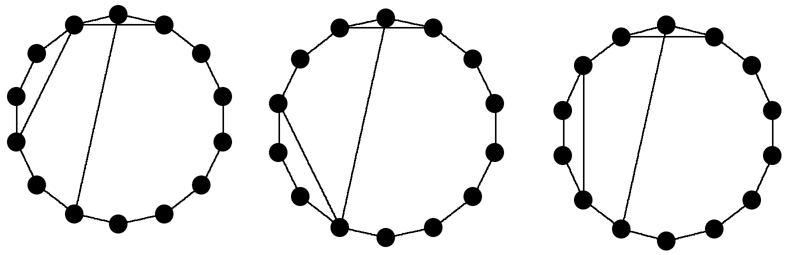

Figure 2: The UPC graphs on 14 vertices

Between [6] and the current paper there appears to have been no further progress on this problem. Steven Locke [5] keeps a record of progress on all the 50 problems in [2].

In the present paper we first use an analysis of the graph's cycle space to give new significantly improved bounds on the size of a UPC graph of a given order. These bounds are sharp for the known UPC graphs, and for the case of UPC graphs with five chords leaves only one possible order.

We then use a computer search in order to first verify that Shi indeed has found all UPC graphs on at most 14 vertices and all UPC graphs with at most 4 chords. This search is extended to show that there are no UPC graphs except those in Figure 1 and Figure 2 with at most 59 vertices.

2 Bounding the number of edges

Since a UPC graph G on n vertices must be hamiltonian it can be constructed by adding chords to a cycle C_n of length n, let k denote the number of chords added. We say that two chords e_1 and e_2 cross each other if $C_n \cup e_1 \cup e_2$ is a K_4-subdivision. Let c denote the number of crossing pairs of chords, let c_3 denote the number of unordered triples of pairwise crossing chords. Let c_Δ be the number of unordered triples $\{e_1, e_2, e_3\}$ of chords such that no pair of chords e_i, e_j cross each other and there is a cycle using all three chords.

We will now give an upper and a lower bound for the number of chords in a UPC graph.

Theorem 2.1. *The number of chords k in a UPC graph G satisfy inequality 2.1, and if G has at least 4 chords inequality 2.2 as well:*

$$3 + 2k + \binom{k}{2} + (k-1)c - c_3 + c_\Delta \quad \leq \quad n \tag{1}$$

$$\log_2(n - 1 + f(k, \Delta)) + \log_2\left(\frac{4}{7}\right) \quad \leq \quad k, \textit{for } k \geq 4 \tag{2}$$

with

$$f(k,\Delta) = \begin{cases} \frac{1}{24}(8k^2 - 32k + 23), & \textit{for } \Delta = 3 \\ 2^{k-2}7\left(1 - \frac{8}{7}(1 + \Delta + \frac{1}{2}\binom{\Delta}{2})\right), & \textit{for } \Delta \geq 4 \end{cases}$$

Proof. **Inequality 1**: For inequality 1 we observe that our graph first contains one hamiltonian cycle, and each chord added determines two cycles by cutting the hamiltonian cycle into two parts, giving us $1 + 2k$ cycles.

Any pair of chords determines at least one cycle, containing both chords, and if the chords cross each other they determine one further cycle, giving us another $\binom{k}{2} + c$ cycles.

Next we will look at cycles using three chords. We get four subcases.

1. A set of three pairwise non-crossing chords. If the three chords are of the type counted by c_Δ they give rise to exactly one cycle, otherwise they do not give rise to a cycle.

2. Two crossing chords together with a chord not crossing any of the first two. In this case there is exactly one cycle using all three chords.

3. One chord crossing two other chords which do not cross each other. In this case there is exactly two cycles using all three chords.

4. Three pairwise crossing chords. In this case there is exactly two cycles using all three chords.

In case (2) and (3) the number of cycles is the same as the number of crossings among the three chords used. However in case (4) we use three crossings and get just two cycles. So by choosing one of the c pairs of crossing chords together with one of the remaining $k-2$ chords we always define a cycle. Taking multiplicities into account we get $c(k-2) - c_3$ cycles involving three chords and at least one crossing, and a further c_Δ cycles from triples of chords without a crossing.

This adds up to $1 + 2k + \binom{k}{2} + c + (k-2)c - c_3 + c_\Delta$ cycles and must be at most $n-2$, the number of cycles in a UPC graph.

Inequality 2: To prove inequality 2 we will make use of the cycle space of our graph. For some basic facts about cycle spaces see e.g. [3]. We first observe that the cycle space of our graph has dimension $k+1$ and thus has 2^{k+1} elements. Some number a of these elements are even subgraphs which are not cycles but rather graphs containing vertices even degrees higher than two and/or several components; for later convenience we will call these graphs a-graphs. Since our graph has $n-2$ cycles and these are members of the cycle space we must have that

$$2^{k+1} - a = n - 2$$

.

Next we will show that a is at least $2^{k-2} + 1 + f(k)$. The "+1" term comes from the empty graph which is a member of the cycle space and so we focus on the $2^{k-2} + f(k)$ contribution. Given a chord e let C_e be the shorter of the two cycles formed by adding e to the hamiltonian cycle in G, if they have the same length C_e can be taken as the cycle containing the edge $(1, n)$.

The proof is now divided into two cases according to the maximum degree of G.

$\Delta(G) \geq 4$ Let v be a vertex of maximum degree in G. Let A be a subset of the chordes incident with v of size at least three. The sum of the cycles C_e for $e \in A$ together with the cycles corresponding to any subets set of the chords not incident with v forms an a-graph, since v will have degree greater than three. Adding the hamiltonian cycle to this a-graph will give us yet another a-graph. A subset of size 2 of the chords incident with v, together with any subets of the chords not incident with v, will give an a-graph either directly or after adding the hamiltonian cycle, since the degree of v in one of these two subgraphs will be at least 4. The number of nontrivial a-graphs is thus at least

$$2\left(2^{\Delta}-1-\delta-\binom{\Delta}{2}\right)2^{k-\Delta}+\binom{\Delta}{2}2^{k-\Delta}=$$
$$=2^{k-2}+2^{k-2}7\left(1-\frac{8}{7}(1+\Delta+\frac{1}{2}\binom{\Delta}{2})\right) \quad (3)$$

$\Delta(G)=3$ In this case there will exist a chord e_1 such that e_1 is the only chord which is part of the triangle in G, call the triangle C_{e_1}, and there might be one further chord e_2 which is incident with the third vertex of the triangle.

Next let A be a nonempty subset of the remaining chords and let $C_A = \sum_{e\in A} C_e$. Now either C_A or $C_A + C_{e_1} + C_n$ will have two components and is an a-graph. If there is no edge e_2 we will here have another $2^{k-1}-1 = 2^{k-2}+2^{k-2}-1$ a-graphs and we are done. If there is a chord e_2 we get 2^{k-2} a-graphs this way.

We now assume that there is a chord e_2. Let n_1 be the number of chords which do not cross e_2. Any such chord together with e_2 give rise to one a-graph with two components, and likewise for any pair of such chords. In the same way they can also combine with both e_1 and e_2. We thus get another $2(n_1+\binom{n_1}{2})$.

Let n_2 be the number of chords, apart from e_1, which cross e_2. Any pair och such chords will create a two component a-graph which uses both chords and both of e_1 and e_2. Let c_1 be the number of such pairs of chords. Any such pair of non crossing chords will create a two component a-graph which uses neither of e_1 and e_2. Let c_2 be the number of such pairs of chords. We thus get another c_1+c_2 a-graphs, with $c_1+c_2=\binom{n_2}{2}$.

If we minimize the expressions for the number of a-graphs under the constraint that $n_1+n_2=k-2$ some simple algebra gives us that the number of a-graphs is at least

$$2^{k-2}+\frac{1}{24}(8k^2-32k+23).$$

We thus have that $2^{k+1}-2^{k-2}-1-f(k)\geq n-2$, with $f(k)$ bounded as promised, and a little algebra gives us 2 □

We first note that in this proof we mainly made use of the fact that our graph is hamiltonian and contains a triangle, as well as the number of cycles and a-graphs in the cycle space, to bound the number of chords and crossings. However the fact that our graphs are hamiltonian is essential. In [4] Lai has proven that the maximum number of edges in a graph will no repeated cycle lengths is asymptotically at least $n+\sqrt{n\left(2+\frac{2}{5}\right)}$, which is greater than that allowed by Inequality 2. In contrast to the UPC graphs, the graphs constructed by Lai have only cycles of length $o(n)$. To determine the exact value of this parameter was suggested in 1975 by Erdös, and is another of the open problems in Bondy and Murty. A natural variation in the current context would be:

Problem 2.2. *Determine the maximum number of edges in a hamiltonian graph on n vertices with no repeated cycle lengths.*

Inequality 1 is sharp for c up to, at least, $3k$ in the sense that for values of c in this range there are hamiltonian non-UPC graphs with exactly this number of cycles, chords, and crossings. The bound is also tight for all of the known UPC graphs. The methods behind Inequality 2 gives sharp bounds for all the known UPC-graphs.

As mentioned in the introduction Shi has proven that there are no further outerplanar UPC graphs and so it is natural to turn our attention to planar UPC graphs in general. For planar graphs we get that $c_3 = 0$, since three edges of the kind counted by c_3 give rise to a subdivided $K_{3,3}$ and excluding a K_5-subdivision as a subgraph will further restrict the cycle space. However we have not found any simple way of excluding further planar UPC graphs.

3 The Computer Search

In order to look for new UPC graphs, and verify the results of Shi regarding UPC graphs with at most 4 chords, an exhaustive computer search was performed. In order to reduce the risk for errors I wrote two separate programs to do this, the first in `Mathematica`, and the second in Fortran 90. The programs worked by starting with a cycle on n vertices and then trying to add chords in a way which did not create repeated cycle lengths. The fortran program also partitioned the search into parts with given maximum degree of the graphs and given length of the shortest chord. This partitioning was done in order to be able to use several computers in the search.

Both programs found all the known UPC graphs and it was also made sure that they found the same partial UPC graphs for small n. The Mathematica program was run as far as possible on using several PowerMac computers and the Fortran 90 program was run on a linux cluster. The Fortran program was stopped when the individual subcases had reached a run time of several weeks, on a machine with a 1.66GHz Athlon processor.

The results of the computer search is as follows:

Observation 3.1. *The graphs in Fig.1 and Fig.2 are the only UPC graphs on $n \leq 59$ vertices.*

Observation 3.2. *There are no UPC graphs with the following number of vertices and maximum degrees. $4 \leq \Delta \leq 6$: $n \leq 60$, $\Delta = 7$: $n \leq 67$, $\Delta = 8$: $n \leq 75$, $\Delta = 9$: $n \leq 81$, $\Delta = 10$: $n \leq 92$, $\Delta = 11 \ldots 15$: $n \leq 100$*

Observation 3.3. *The graphs of Figure 1 and Figure 2 are the only UPC graphs with at most 5 chords.*

Proof. Inequality 2 gives an upper bound of 53 vertices for UPC graphs with at most 5 chords and any such graphs beyond those already known are ruled out by 3.1. □

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