Two questions of Erdős on hypergraphs above the Turán threshold

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Abstract

For ordinary graphs it is known that any graph G with more edges than the Turán number of K_s must contain several copies of K_s, and a copy of K_{s+1}^-, the complete graph on $s+1$ vertices with one missing edge. Erdős asked if the same result is true for K_s^3, the complete 3-uniform hypergraph on s vertices.

In this note we show that for small values of n, the number of vertices in G, the answer is negative for $s = 4$. For the second property, that of containing a ${K_{s+1}^3}^-$, we show that for $s = 4$ the answer is negative for all large n as well, by proving that the Turán density of K_5^{3-} is greater than that of K_4^3.

1 Introduction

One of the cornerstones of modern graph theory is Turán's theorem, which gives the maximum number of edges in a graph on n vertices with no complete subgraphs of order s. This theorem has been extended in a number of different directions and we now have a good understanding of the corresponding question for other forbidden subgraphs, as long as they are not bipartite, and the structure of graphs close to the Turán threshold.

Recall that the Turán number $t(n, s, k)$ is the maximum number of edges in a k-uniform hypergraph on n vertices which does not have the complete k-uniform hypergraph on s vertices as a subgraph. Rademacher proved, but did not publish, see [Erd62] that any graph with $t(n, 3, 2) + 1$ edges contains at least $n/2$ triangles. Erdős later [Erd62] published a proof of this result and extended it by proving that for $0 \leq q \leq c_1 n/2$, for some constant c_1, any graph with $t(n, 3, 2) + q$ edges contains at least $qn/2$ triangles. This result was later extended to graphs with density higher than $t(n, 3, 2)/\binom{n}{2}$, recently culminating in [Raz08] where the minimal number of triangles was determined for all densities.

Erdős and Dirac also observed that any graph with $t(n, s, 2) + 1$ edges has K_s^-, the complete graph with one edge removed, as a subgraph. This provides a strengthening of the result for triangles from [Erd62] in the sense that it shows that the graph contains two copies of K_3 which share two vertices.

In [Erd94] Erdős asked if these results could be generalized to 3-uniform hypergraphs as well

Problem 1.1. *Does every 3-uniform hypergraph on $t(n, s, 3) + 1$ edges contain two K_s^3?*

Problem 1.2. *Does every 3-uniform hypergraph on $t(n, s, 3) + 1$ edges contain K_{s+1}^{3-}?*

In [CG98] these questions, with an affirmative answer to both, were formulated as conjectures,

The aim of this note is to show that the answer to both questions is no. For the first question we believe that the answer is yes for sufficiently large n, but as we shall see the second question fails even in an asymptotic sense.

2 The examples

We first construct a family of hypergraphs which give a negative answer to the two questions for certain small values of n.

Example 2.1. The k-uniform hypergraph H_k is the hypergraph with vertex set $V = [2k - 1]$ and edge set $E = \binom{[2k-1]}{k} \setminus \{\{1, 2, \ldots, k\}, \{1, k + 1, k + 2, \ldots, 2k - 1\}\}$

We now have

Theorem 2.2. *Let H_k be define as in the example.*

1. *H_k does not contain two K_{2k-2}^k*
2. *$|E(H_k)| \geq t(2k - 1, 2k - 2, k) + 1$*

Proof.

1. The vertex set of any K_{2k-2}^k in H_k cannot have one of the two non-edges as a subset. Let A and B be subsets of $V(H_k)$ of size $2k-2$. Since any vertex subset of size $2k - 2$ misses one vertex of H_k at least one of A and B must have one of the two non-edges as a subset.

2. Since every subset of $V(H_kj)$ of size $2k-2$ can have at most $\binom{2k-2}{k} - 1$ edges, we find by averaging that

$$t(2k - 1, 2k - 2, k) \leq \frac{\binom{2k-2}{k} - 1}{\binom{2k-2}{k-2}} \binom{2k - 1}{k} = \binom{2k - 1}{k} - \frac{2k - 1}{k - 1} \quad (1)$$

The number of edges in H_k is $\binom{2k-1}{k} - 2$ and Inequality 1 shows that, since $\frac{2k-1}{k-1} > 2$, the Turán number $t(2k - 1, 2k - 2, k)$ is at most $\binom{2k-1}{k} - 3$.

□

Corollary 2.3. *H_3 gives a negative answer to Problem 1.1 and since there are two non-edges in H_k it does not contain a K_5^{3-} either, thereby giving a negative answer to Problem 1.2 as well.*

Note that H_k also provides a negative answer to a generalisation of both problems to k-graphs for $k \geq 3$. However we believe that Question 1 should have a positive answer for sufficiently large values of n

Conjecture 2.4. *There is an $n_0(s)$ such that every 3-uniform hypergraph on $n \geq n_0(s)$ vertices and $t(n, s, 3) + 1$ edges contains two K_s^3*

For problem 1.2 the failure is not a small n phenomenon, as our next theorem will imply. Recall that the Turán density $\pi(G)$ of a k-uniform hypergraph G is

$$\pi(G) = \lim_{n\to\infty} \frac{t(n, G)}{\binom{n}{k}},$$

where $t(n, G)$ is the maximum number of edges in a k-uniform hypergraph on n vertices which does not have G as a subgraph. We note that a positive answer to Problem 1.2 requires that $\pi(K_s^3) = \pi(K_{s+1}^{3-})$. For K_5^{3-} we have the following lower bound,

Theorem 2.5. $\pi(K_5^{3-}) \geq \frac{46}{81}$

Proof. Let H_3 be the 3-uniform hypergraph defined in our earlier example and let $H_3(n)$, for n divisible by 9, be the blow-up of H_3 with $n/9$ copies of vertex 1 in H_3 and $2n/9$ copies of each of the other vertices in H_3. Three vertices v_i, u_j, w_k form an edge in $H_3(n)$ if $\{v, u, w\}$ is an edge in H_3. It is easy to see that $H_3(n)$ does not contain a K_5^{3-} and a simple calculation shows that the number of edges in $H_3(n)$ is $(\frac{32}{81} + o(1))\binom{n}{3}$

Next, let D be the directed graph on the same vertex set as H_3 shown in Figure 1. We now define a 3-uniform hypergraph $G_0(n)$ on the same

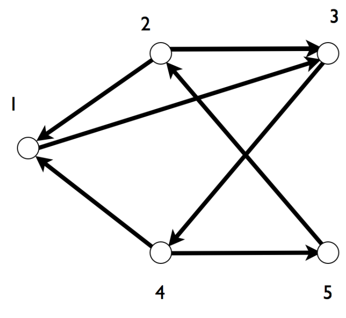

Figure 1: The auxiliary graph D

vertex set as $H_3(n)$ where $\{v_i, v_j, u_k\}$ is an edge if, $i < j$ and there is an edge from v to u in D. An edge in D not incident with vertex 1 gives rise to $(\frac{8}{243}+o(1))\binom{n}{3}$ edges. An edge leading to vertex 1 gives rise to $(\frac{4}{243}+o(1))\binom{n}{3}$ edges, and an edge leading from vertex 1 gives rise to $(\frac{2}{243}+o(1))\binom{n}{3}$ edges.

Finally we let $G(n)$ be the graph on the same vertex set as $H_3(n)$ with all edges from $H_3(n)$ and $G_0(n)$. The number of edges in $G(n)$ is

$$(\frac{32}{81}+o(1))\binom{n}{3}+(\frac{2}{243}+o(1))\binom{n}{3}+2(\frac{4}{243}+o(1))\binom{n}{3}+ \\ 4(\frac{8}{243}+o(1))\binom{n}{3}=(\frac{46}{81}+o(1))\binom{n}{3} \quad (2)$$

In order to prove the theorem we now need to show that $G(n)$ does not contain K_5^{3-}.

Let A be a set of 5 vertices in $G(n)$. If A corresponds to five distinct vertices in H_3 then A is not a K_5^{3-}. If A contains four or five vertices corresponding to the same vertex in H_3 then there are at least 4 non-edges in A, and A is not a K_5^{3-}. If A has two vertices corresponding to the same vertex in H_3 and two other vertices corresponding to another vertex in H_3 then there are at least 2 non-edges in A, and A is not a K_5^{3-}.

There are now two remaining possibilities:

1. There are three vertices in A corresponding to the same vertex v in H_3 and two vertices corresponding to two other, distinct, vertices u and w in D. If there is at most one edge from v to u and w in D then there are at least three non-edges in A, and A is not a K_5^{3-}. If there are two edges from v to u and w then $\{v, u, w\}$ must be one of the two non-edges in H_3, which means that there are at least three non-edges in A, and A is not a K_5^{3-}.

2. There are two vertices in A corresponding to the same vertex v in H_3 and three vertices corresponding to three other, distinct, vertices u, w and z in D. If there are two edges from v to u, w and z then v together with two of the other vertices are one of the non-edges in H_3 and there are at least two non-edges in A, and A is not a K_5^{3-}. If there is only one edge from v to the other three vertices there are at least two non-edges in A, and A is not a K_5^{3-}.

Hence $t(n, K_5^{3-}) \geq (\frac{46}{81}+o(1))\binom{n}{3}$ and $\pi(K_5^{3-}) \geq \frac{46}{81}$.

□

Turán [Tur61] posed the problem of finding the Turán density of the complete uniform hypergraphs and conjectured that $\pi(K_4^3) = \frac{5}{9}$, and gave a construction attaining this value. The conjecture remains open and the best current upper bound has been found by Razborov.

Lemma 2.6. *[Raz10]* $\pi(K_4^3) \leq 0.561666$

The derivation of this bound in [Raz10] is done by the flag-algebra method and is semi-numerical, but there is now freely available software [BT11] which will give a computer assisted but non-numerical proof, based on the flag-algebra method.

Our lower bound for $\pi(K_5^{3-})$ is $\frac{46}{81} = 0.5679\ldots$, which is larger than the upper bound for $\pi(K_4^3)$. Hence we get our desired corollary,

Corollary 2.7. *For $s = 4$ Problem 1.2 has a negative answer for all sufficiently large n*

We believe that similar constructions will be possible for small values of s and make the following conjecture,

Conjecture 2.8. $\pi(K_s^3) < \pi(K_{s+1}^{3-})$*, for* $s = 5, 6$

However, for large s the two densities might coincide. In particular it would be interesting to settle the following question

Problem 2.9. *Is* $\pi(K_7^3) = \pi(K_8^{3-})$*?*

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