Estimation of the capacity of multidimensional constraints using the 1-vertex transfer matrix

S. Friedland, P.H. Lundow and K. Markström

***Abstract*—The notion of a 1-vertex transfer matrix for multi-dimensional codes is introduced. It is shown that the capacity of such codes, or the topological entropy, can be expressed as the limit of the logarithm of spectral radii of 1-vertex transfer matrices. Storage and computations using the 1-vertex transfer matrix are much smaller than storage and computations needed for the standard transfer matrix.**

The method is applied to estimate the first 15 digits of the entropy of the 2-dimensional (0,1) run length limited channel.

A large scale computation of eigenvalues for the (0,1) run length limited channel in 2 and 3 dimensions have been carried out. This was done in order to be able to compare the computational cost of the new method with the standard transfer matrix and have rigorous bounds to compare the estimates with. This in turn leads to improvements on the best previous lower and upper bounds for these channels.

2000 Mathematics Subject 2000 Mathematics Subject Classification: 05A16, 28D20, 37M25, 82B20

Keywords and phrases: Channel capacity, transfer matrices, multi-dimensional codes, optical storage.

I. Introduction

In classical information theory the emphasis was on one-dimensional channels. This is the natural channel type when one is considering sequential information transfer and information storage on effectively one-dimensional media, such as a magnetic tape. However, in modern information technology higher dimensional channels are becoming more and more important. The simplest example is of course information storage on a surface, as on a CD or DVD. Future devices seem likely to move on to three-dimensional storage media as well.

Just as for the classical one-dimensional channels one still wishes to construct codes to protect data from errors and to find the channel capacity given by the code. In higher dimensions computing the exact capacity becomes much more difficult and even for very simple codes we do not know the exact value. The typical example is the run-length limited channel. Consider a two-dimensional square grid where each point can be assigned a value of either 0 or 1 with the restriction that two points which differ by one in exactly one coordinate cannot both be 1. This channel has been studied in both information theory [1], combinatorics [2] and physics [3]. In statistical physics this has been studied in terms of lattices gases, primarily through non-rigorous methods and simulation. However a sequence of authors [2], [4], [5], [1], [6] have developed methods for establishing rigorous upper and lower bounds by using transfer matrices. The drawback with these methods is that the cost of achieving a given precision is exponential in terms of the precision. It is possible to use symmetries [7] to speed up the computation but typically the cost remains exponential, with improved constants.

Hence it would be desirable to be able to establish good estimates for the capacity of a code before deciding whether to invest the time needed to bound its capacity rigorously. In this paper we present a method for obtaining such estimates. The method uses a so called 1-vertex transfer matrix which adds one vertex at a time to the graph, rather than whole rows as in the standard method. We prove that under certain conditions on the code the largest eigenvalue of this matrix converges to the channel capacity. The drawback is that we do not get easily computable bounds for the capacity, but as a practical example will demonstrate one can get estimates which agrees with the rigorous bounds at a fraction of the computational cost. Hence this method can be used as a tool for quickly testing many different candidate codes in order to identify promising coding schemes, for which one can later establish rigorous bounds as well.

In order to help evaluate the accuracy of our new method we have also improved the known bounds for the capacity of the two and three-dimensional run-length limited channels, or hard core-lattice gases as they are known in the physics literature.

A. Mathematical preliminaries

Many channels and models from statistical physics can be described in terms of restricted colourings of an underlying graph, often a lattice such as $\mathbb{Z}^d$, or more generally in terms of weighted graph homomorphisms [8], [9], [7].

We now explain the main ideas and results of our paper for the two-dimensional grid $\mathbb{Z}^2$, where the colouring conditions are *symmetric* and *isotropic*. Let $\Delta = (V, E), V = \{1, \ldots, k\}$ be an undirected graph on k vertices, where vertex i is identified with the colour i. Two neighbouring points $\mathbf{p} = (p_1, p_2)^\top, \mathbf{q} = (q_1, q_2)^\top \in \mathbb{Z}^2$, i.e. $|p_1 - q_1| + |p_2 - q_2| = 1$, can be coloured in colours $i, j \in V$ if and only if $(i, j) \in E$. We assume that Δ has no isolated vertices.

The standard transfer matrix is defined as follows. For a positive integer $n \in \mathbb{N}$ denote $\langle n \rangle = \{1, \ldots, n\}$. Then $\phi : \langle n \rangle \to V$ is viewed as a colouring of $\langle n \rangle$ in k colours. View $\langle n \rangle$ as n points on the real line, where $i, j \in \langle n \rangle$ are neighbours if and only if $|i - j| = 1$. Then $\phi \in \mathrm{C}_\Delta(\langle n \rangle)$

Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, Illinois 60607-7045, USA, <friedlan@uic.edu>.

Department of Theoretical Physics, AlbaNova University Center, KTH, SE-106 91 Stockholm, Sweden, <phl@kth.se>.

Department of Mathematics and Mathematical Statistics, UmeåUniversity, SE-901 87 Umeå, Sweden, <Klas.Markstrom@math.umu.se>.

is an allowable colouring if $(\phi(i), \phi(j)) \in E$ whenever $|i-j| = 1$. Let $N(n) = |\mathrm{C}_\Delta(\langle n \rangle)|$ be the cardinality on $\mathrm{C}_\Delta(\langle n \rangle)$. Note that $N(n) = \Theta(\rho(\Delta)^n)$, where $\rho(\Delta)$ is the spectral radius of Δ. Let $\phi, \psi \in \mathrm{C}_\Delta(\langle n \rangle)$ be two allowable Δ colourings of $\langle n \rangle$. View (ϕ, ψ) as a colourings of integer strip $\tau : \langle n \rangle \times \langle 2 \rangle \subset \mathbb{Z}^2$, where $\tau((i,1)) = \phi(i), \tau((i,2)) = \psi(i)$ for $i = 1, \ldots, n$. Let $t_{\phi,\psi} = 1$ if τ is a Δ-allowed colouring, i.e. $(\phi(i), \psi(i)) \in E$ for $i = 1, \ldots, n$, and otherwise $t_{\phi,\psi} = 0$. Then $T_{n,2} = [t_{\phi,\psi}]_{\phi,\psi \in \mathrm{C}_\Delta(\langle n \rangle)} \in \{0,1\}^{N(n) \times N(n)}$ is the transfer matrix, which defines the allowable colourings of the infinite strip $\langle n \rangle \times \mathbb{Z} \subset \mathbb{Z}^2$. An important and useful observation is that $T_{n,2}$ is a symmetric matrix.

Let $\rho(T_{n,2})$ be the spectral radius of $T_{n,2}$. It is well known that the sequence $\log \rho(T_{n,2})$ is *subadditive* in n. Hence the following limit exists, and it is called the *topological entropy* of $\mathbb{Z}^2$ with respect to Δ, e.g. [10], [11], [5]:

$$h_2(\Delta) := \lim_{n \to \infty} \frac{\log \rho(T_{n,2})}{n}. \tag{I.1}$$

Furthermore, one has the following known upper and lower bounds, see e.g. [2], [10]

$$\frac{\log \rho(T_{p+2q+1,2}) - \log \rho(T_{2q+1,2})}{p} \le h_2(\Delta) \le \frac{\log \rho(T_{n,2})}{n}, \quad 1 \le p, n, \ 0 \le q. \tag{I.2}$$

In many theoretical derivations, e.g. [12], it is useful to consider the periodic colouring $\mathrm{C}_{\Delta,\mathrm{per}}(\langle n \rangle) \subset \mathrm{C}_\Delta(\langle n \rangle)$, where each colouring $\phi \in \mathrm{C}_\Delta(\langle n \rangle)$ satisfies the additional condition $(\phi(1), \phi(n)) \in E$. So $\mathrm{C}_{\Delta,\mathrm{per}}(\langle n \rangle)$ is a Δ-allowable colouring of the torus $\mathbb{Z}/\langle n \rangle$. Let $N(n, \mathrm{per}) = |\mathrm{C}_{\Delta,\mathrm{per}}(\langle n \rangle)|$ be the number of periodic Δ-colouring of $\mathrm{C}_{\Delta,\mathrm{per}}(\langle n \rangle)$. Then $T_{n,2,\mathrm{per}} = [t_{\phi,\psi}]_{\phi,\psi \in \mathrm{C}_{\Delta,\mathrm{per}}(\langle n \rangle)} \in \{0,1\}^{N(n,\mathrm{per}) \times N(n,\mathrm{per})}$ is the transfer matrix for the allowable colourings of the infinite torus $(\mathbb{Z}/\langle n \rangle) \times \mathbb{Z}$ by Δ. Note that $T_{n,2,\mathrm{per}}$ is a principal submatrix of $T_{n,2}$, hence symmetric. Furthermore $\rho(T_{n,2,\mathrm{per}}) \le \rho(T_{n,2})$. Also, the following inequalities hold [2]

$$\frac{\log \rho(T_{p+2q,2,\mathrm{per}}) - \log \rho(T_{2q,2,\mathrm{per}})}{p} \le h_2(\Delta) \le \frac{\log \rho(T_{2n,2,\mathrm{per}})}{2n}, 0 \le q, \ 1 \le p, n, \tag{I.3}$$

where $\rho(T_{0,2,\mathrm{per}}) = \rho(\Delta)$. Note that the upper bound in (I.3) is an improvement of the upper bound in (I.2) for an even n.

We now define the 1-vertex transfer matrix

$$S_{n,2} = [s_{\phi,\psi}]_{\phi,\psi \in \mathrm{C}_\Delta(\langle n \rangle)} \in \{0,1\}^{N(n) \times N(n)}, \tag{I.4}$$

where

$$s_{\phi,\psi} = 1 \text{ iff } \psi(i) = \phi(i+1) \text{ for } i = 1, \ldots, n-1, \text{ and } (\phi(1), \psi(n)) \in E. \tag{I.5}$$

$S_{n,2}$ is the transfer matrix corresponding to the Δ colouring of $\mathbb{Z}$ winded up on a *slanted* torus, given by $\langle n+1 \rangle \times \mathbb{Z}$, where the points $(n+1, i)$ and $(1, i+1)$ are identified for each $i \in \mathbb{Z}$, see Figure I-A. Note that $S_{n,2}$ is a sparse matrix, where each row and column of $S_{n,2}$ has at most k neighbours. We show that

$$\lim_{n \to \infty} \log \rho(S_{2n+1,2}) = h_2(\Delta). \tag{I.6}$$

Thus $\log \rho(S_{n,2})$ is an approximation of $h_2(\Delta)$, and it can be computed much faster, and with much less memory than $\log \rho(T_{n,2})$ and $\log \rho(T_{n,2,\mathrm{per}})$. The only drawback for computing the values of $\log \rho(S_{n,2})$, is that we do not have very sharp inequalities of the type (I.2) or (I.3). One of the reasons is that $S_{n,2}$ is not a symmetric matrix and can have nonreal eigenvalues. We do have the following inequalities

$$\frac{\log \rho(T_{n-1,2,\mathrm{per})})}{n} \le \log \rho(S_{n,2}) \le \min\left(\frac{\log \rho(T_{n,2})}{n}, \frac{\log \rho(T_{n+1,2,\mathrm{per}})}{n}\right), \ 2 \le n, \tag{I.7}$$

which imply (I.6).

Fig. 1. An example of the vertex labeling used in the defition of the 1-vertex transfer matrix

The undirected graph Φ, which can be viewed as a symmetric digraph on two vertices

$$\Phi = (V, E), \quad V = \{1, 2\}, \ E = \{(1,2), (2,1), (2,2)\}, \tag{I.8}$$

corresponds to the checkerboard constraint used in [6] and hard core lattice gas model in statistical mechanics. (Usually 2 is replaced by 0.) We will use the 1-vertex transfer matrix to give a heuristic estimate of the entropy of the two dimensional hard core $h_2(\Phi)$ up to 15 digits. Using the upper and lower bounds (I.2) and (I.3) we show that the 15 digits estimates agree with the correct values of the $h_2(\Phi)$. Previously the value of $h_2(\Phi)$ was known to 10 digits [1].

B. Overview

We now briefly survey the contents of the paper. In §2 we give the full details of the two-dimensional isotropic symmetric case discussed above. In §3 we discuss the computational results for the two-dimensional checkerboard constraint/hard core lattice gas. In §4 we discuss the characteristic polynomial and the eigenvalues of $S_{n,2}$ for the hard core model. In §5 we discuss a two-dimensional 1-vertex transfer matrix $P_{n,2}$ for nonisotropic and nonsymmetric colouring of $\mathbb{Z}^2$. We show that $\log \rho(P_{n,2})$ is an estimate of the two dimensional-capacity, or entropy, if the colouring of $\mathbb{Z}^2$ has a friendly colour. In §6 we show that the results for two dimensions apply also to the three-dimensional 1-vertex matrix. In the last section we show that 1-vertex model can also be used to estimate the monomer-dimer model in any dimension d.

II. The Two-dimensional isotropic symmetric case

Recall the graph $\Delta = (V, E)$, where the set of vertices V is identified with k colours $\langle k\rangle$. We allow loops $(i,i) \in E$, which means that two adjacent vertices of $\mathbb{Z}^2$ can be coloured with the same colour i. If $i \in V$ is an isolated vertex, this means that no vertex in $\mathbb{Z}^2$ can be coloured in the colour i. Hence we will deal only with graphs Δ which do not have isolated vertices.

Theorem 2.1: Let $\Delta = (V, E)$ be an undirected graph on $|V| = k \ge 2$ vertices, where self-loops are allowed, with no isolated vertices. Then (I.6) and (I.7) hold.

Proof: Identify the entry $s_{\phi,\psi} = 1$ of $S_{n,2}$ with an allowable Δ-colouring of the following L shape in $\mathbb{Z}^2$

$$\begin{aligned} &\omega : \{(1,1), \ldots, (n,1), (1,2)\} \to \langle k\rangle, && \text{(II.1)}\\ &\omega((i,1)) = \phi(i),\ i = 1, \ldots, n,\ \omega((1,2)) = \psi(n). \end{aligned}$$

More generally, consider the entries of the matrix $S_{n,2}^m = [s_{\phi,\psi}^{(m)}]_{\phi,\psi\in \mathrm{C}_\Delta(\langle n\rangle)}$. Recall that

$$s_{\phi_1,\phi_{m+1}}^{(m)} = \sum_{\phi_1,\ldots,\phi_{m+1}\in \mathrm{C}_\Delta(\langle n+1\rangle)} \prod_{i=1}^m s_{\phi_i,\phi_{i+1}}.$$

Clearly, $\prod_{i=1}^m s_{\phi_i,\phi_{i+1}} \in \{0,1\}$. Then $\prod_{i=1}^m s_{\phi_i,\phi_{i+1}} = 1$ if and only if one has a Δ-allowable colouring $\tau \in \mathrm{C}_\Delta(\langle n+m\rangle)$ such that

$$(\tau(i), \tau(j)) \in E \text{ if } |i-j| = n. \qquad \text{(II.2)}$$

Assume that $\prod_{i=1}^m s_{\phi_i,\phi_{i+1}} = 1$. Then we can colour the following part of $\langle n\rangle \times \mathbb{Z}$:

$$\begin{aligned} &(1,1), \ldots, (n,1); (1,2), \ldots, (1,q); \ldots; \ldots; (1,q) \ldots, (l,q),\\ &\qquad q = \left\lceil \frac{n+m}{n} \right\rceil,\ l = n+m-(q-1)n, \end{aligned}$$

using τ. Let $\mathbf{1} = (1, \ldots, 1)^\top$ be a vector whose all coordinates are 1. The number of coordinates of $\mathbf{1}$ will depend on the square matrix $A = [a_{ij}] \in \mathbb{R}^{M\times M}$. Observe that $\mathbf{1}^\top A\mathbf{1} = \sum_{i,j=1}^M a_{ij}$. Hence $\mathbf{1}^\top S_{n,2}^m \mathbf{1}$ is the number of all Δ allowable colourings $\tau \in \mathrm{C}_\Delta(\langle n+m\rangle)$ satisfying the condition (II.2). In particular, $\mathbf{1}^\top S_{n,2}^{(q-1)n}\mathbf{1}$ is the number of different colourings of the rectangle $\langle n\rangle \times \langle q\rangle$ induced by $\tau \in \mathrm{C}_\Delta(\langle nq\rangle)$ satisfying (II.2). Similarly, $\mathbf{1}^\top T_{n,2}^{q-1}\mathbf{1}$ is the number of Δ-allowable colourings of the strip $\langle n\rangle \times \langle q\rangle$. Hence

$$\begin{aligned} &\mathbf{1}^\top S_{n,2}^{(q-1)n}\mathbf{1} \le \mathbf{1}^\top T_{n,2}^{(q-1)}\mathbf{1} \Rightarrow\\ &\log\rho(S_{n,2}) = \lim_{q\to\infty} \frac{\log \mathbf{1}^\top S_{n,2}^{(q-1)n}\mathbf{1}}{(q-1)n} \le\\ &\lim_{q\to\infty} \frac{\mathbf{1}^\top T_{n,2}^{(q-1)}\mathbf{1}}{(q-1)n} = \frac{\log\rho(T_{n,2})}{n}. \end{aligned}$$

See [11, Proposition 10.1] for details. This inequality establishes the first part of the upper bound in (I.7). The equality (I.1) yields the inequality

$$\limsup_{n\to\infty} \log\rho(S_{n,2}) \le h_2(\Delta). \qquad \text{(II.3)}$$

We now show the inequality $\log\rho(S_{n,2}) \le \frac{\log\rho(T_{n+1,2,\mathrm{per}})}{n}$. Let $m = (q-1)n+1$ and consider the colouring of $\langle n\rangle \times \langle q\rangle \cup \{(1, q+1)\}$ induced by colouring $\tau \in \mathrm{C}_\Delta(\langle nq+1\rangle)$ satisfying condition (II.2). Note that the number of such colourings is $\mathbf{1}^\top S_{n,2}^{(q-1)n+1}\mathbf{1}$. We now extend each colouring τ to the colouring of $\langle n+1\rangle \times \langle q\rangle$ viewed as a colouring of the torus $(\mathbb{Z}/(n+1)) \times \langle q\rangle$ as follows. The colouring of $\langle n\rangle \times \langle q\rangle$ is given by τ as above. Then the colour of the point $(n+1, i)$ is given by $\tau(in+1)$ for $i = 1, \ldots, q$. Hence $\mathbf{1}^\top S_{n,2}^{(q-1)n+1}\mathbf{1} \le \mathbf{1}^\top T_{n+1,2,\mathrm{per}}^{q-1}\mathbf{1}$. Use [11, Proposition 10.1] as above to deduce the second part of the upper bound in (I.7).

We now show the lower bound in (I.7). Recall that $\mathbf{1}^\top T_{n-1,2,\mathrm{per}}^{q-1}\mathbf{1}$ is the number of colourings of the rectangle $\omega : \langle n-1\rangle \times \langle q\rangle$ on the torus $(\mathbb{Z}/(n-1)) \times \langle q\rangle$. Each ω induces the following $\tau \in \mathrm{C}_\Delta(\langle nq\rangle)$ satisfying (II.2):

$$\begin{aligned} &\tau(i + (j-1)n) = \omega((i,j)),\ \tau(jn) = \omega((1,j)),\\ &\qquad\qquad i = 1, \ldots, n-1,\ j = 1, \ldots, q. \end{aligned}$$

Hence $\mathbf{1}^\top T_{n-1,2,\mathrm{per}}^{q-1}\mathbf{1} \le \mathbf{1}^\top S_{n,2}^{(q-1)n}\mathbf{1}$. The above arguments show the lower inequality in (I.7).

Combine the upper bound in (I.3) and the lower bound in (I.7) to obtain

$$\frac{2n}{2n+1} h_2(\Delta) \le \frac{\log\rho(T_{2n,2,\mathrm{per}})}{2n+1} \le \log\rho(S_{2n+1,2}). \qquad \text{(II.4)}$$

Hence $\liminf_{n\to\infty} \log\rho(S_{2n+1,2}) \ge h_2(\Delta)$. Combine this inequality with (II.3) to deduce (I.6). □

III. The Two dimensional run length limited channel

We will now apply the above results to the following simple symmetric digraph Φ on two vertices:

Identify the white colour with the state 1 and the black colour with the state 2, which is usually identified with the state 0. Then $\mathrm{C}_\Phi(\mathbb{Z}^d)$ consists of all colourings of the lattice $\mathbb{Z}^d$ in black and white colours such that no two white colours are adjacent. In terms of codes this corresponds to the checkerboard constraint used in e.g. [1] and the simplest *hard core model* in statistical mechanics.

The adjacency matrix $A(\Phi)$ is $\begin{bmatrix} 0 & 1\\ 1 & 1 \end{bmatrix}$. It is well known that the one-dimensional entropy corresponding to $\mathrm{C}_\Phi(\mathbb{Z})$ is the logarithm of the golden ratio $\frac{1+\sqrt{5}}{2}$. Hence, sometimes Φ is referred as the golden ratio graph.

A. Computational results

The topological entropy $h_2(\Delta)$ of the hard core model is known within the precision of 10 digits [1]. As in the physics literature, e.g. [12], we will discuss the values of $e^{h_2(\Phi)}$. The inequalities obtained in [1] are equivalent to

$$\begin{aligned} &1.503048082474977458599857 34\\ &< e^{h_2(\Phi)} < \qquad\qquad \text{(III.1)}\\ &1.5030480825718679778800 4159 \end{aligned}$$

n	$\rho(T_{n,2})$	$\rho(T_{n,2,\mathrm{per}})$
2	2.414213562373095	
3	3.631381260403638	3.302775637731994646559610633735247
4	5.457705395965834	5.156325174658661693523159039366916
5	8.203259193755024	7.637519478750677316156696280583774
6	12.32988221531524	11.55170956604814509016646221019832
7	18.53240737754881	17.31622927332784947478739705217656
8	27.85509909631079	26.05798609193972135567942994470689
9	41.8675533182809	39.14578184202813825907509993927013
10	62.928945725187815984970517564242	58.85193508152278064182392832406795
11	94.585231204973665631062351227180	88.44780432952028084071406736758034
12	142.16615039284113705381555339180	132.9477940474849517182393096863462
13	213.68255974084561463042598863826	199.8224640440179428924580367714202
14	321.17516167688358891589859286791	300.3458520273548324890314287157792
15	482.74171089714185369639722005855	451.4321236042340748832899060864991
16	725.58400289480913382756934371728	678.5256693463314967782756777455377
17	1090.5876442258184943798276273781	1019.855674771119057225300394186046
18	1639.2056674249062545759683245949	1532.892835974578082578218246454006
19	2463.8049352057025354985027848902	2304.011134761604258032847423262351
20	3703.2172834541914402106141013810	3463.039870272410387224398919252810
21	5566.1136368853314325759144845043	5205.115189611387576190086821772236
22	8366.1364287602952611690001026376	7823.538578192028601159488379441279
23	12574.705316975186002047579953615	11759.15453623673826922032891077820
24	18900.386714371918513370098148691	17674.57476300121240825274824636313
25	28408.190009078957569653791590282	26565.73564566649640784345844846860
26	42698.875519741019043924030420844	39929.57806437579207220381471608921
27	64178.462973799644995496644648926	60016.07571363818523121568140477488
28	96463.315708983666788807112233379	90207.04754031656425802350521860206
29		135585.5298162229950461965077280634
30		203791.5706122926277969020818385026
31		306308.5294259292544744183247215849
32		460396.4478048022575136612396701088
33		691997.9980476929053415088280918978
34		1040106.264045025628338539352181110
35		1563329.725741562336590052426628794
36		2349759.746553886953259135919605940

TABLE I
SPECTRAL RADII FOR STANDARD AND PERIODIC TRANSFER MATRICES

We have computed eigenvalues for the standard transfer matrix, the periodic transfer matrix and finally for the 1-vertex transfer matrix. Let us examine the results.

Table I gives the computed values of $\rho(T_{n,2}), n = 2, \ldots, 28$ and $\rho(T_{n,2,\mathrm{per}}), n = 3, \ldots, 36$. We observed the following facts on the two computed sequences. First the sequence $\frac{\rho(T_{n,2})}{\rho(T_{n-1,2})}$ is an increasing sequence for $n \geq 10$. Note that in view of the lower bound given in (I.2) we know that $\frac{\rho(T_{2m,2})}{\rho(T_{2m-1,2})}$ is a lower bound for $e^{h_2(\Phi)}$. Hence

$$1.50304808247533226432204921 \leq e^{h_2(\Phi)}. \qquad \text{(III.2)}$$

The sequence $\rho(T_{2m,2,\mathrm{per}})^{\frac{1}{2m}}$ is decreasing for $m = 2, \ldots, 18$. In view of the upper bound in (I.3) it follows that

$$e^{h_2(\Phi)} \leq 1.50304808247533992728837255. \qquad \text{(III.3)}$$

The above two inequalities give the correct 15 digits of the value of $e^{h_2(\Phi)}$. Note that the lower bound in (III.1) is closer to the correct value of $e^{h_2(\Phi)}$ than the upper bound. It is plausible to assume that the same observation applies to the inequalities (III.2-III.3).

The sequence $\rho(T_{2m+1,2,\mathrm{per}})^{\frac{1}{2m+1}}$ increases for $m = 1, \ldots, 17$. If this sequence always increases, in view of (I.2), it would follows that $\rho(T_{2m+1,2,\mathrm{per}})^{\frac{1}{2m+1}}$ is a lower bound for $e^{h_2(\Phi)}$. The lower bound given by (III.2) is bigger than $\rho(T_{35,2,\mathrm{per}})^{\frac{1}{35}}$.

n	$\rho(S_{n,2})$
Even n	
26	1.5030480824559338746449982720899
28	1.5030480824713491171046098760579
30	1.5030480824745080695008293589330
32	1.5030480824751605743865692042299
34	1.5030480824752962878823092158144
36	1.5030480824753246862738777999703
38	1.5030480824753306606437859142329
40	1.5030480824753319235292607404167
Odd n	
39	1.5030480824753330032275278142102
37	1.5030480824753357484850986619224
35	1.5030480824753487657242129983806
33	1.5030480824754108025759894900493
31	1.5030480824757081424841278582465
29	1.5030480824771425174857112752302
27	1.5030480824841133358901685021830
25	1.5030480825182810708944214989118

TABLE II
SPECTRAL RADII FOR 1-VERTEX TRANSFER MATRICES

Table II gives the values of $\rho(S_{n,2})$ for $n = 25, \ldots, 40$. We found that the sequence $\rho(S_{2l,2}), l = 1, \ldots, 20$ increases while the sequence $\rho(S_{2l-1,2})$ decreases for $l = 2, \ldots, 20$. Assuming this behaviour for all values $l \in \mathbb{N}$ we deduce that

$$\rho(S_{2l,2}) < e^{h_2(\Phi)} < \rho(S_{2l-1,2}), \text{ for all } l \in \mathbb{N}. \qquad \text{(III.4)}$$

The above *ansatz* for $l = 20$ gives the value of $e^{h_2(\Phi)}$ with a precision of 15 digits, which coincides with the exact values given by (III.2-III.3). More precisely the heuristic upper bound $\rho(S_{39,2})$ is slightly better than (III.3), and the heuristic lower bound given by $\rho(S_{40,2})$ is slightly worse than (III.2).

B. Computational costs

We now compare the computational resources needed in order to compute the eigenvalues for the 1-vertex transfer matrix and the standard transfer matrix for cycles. The computations were done in 31 digits precision, both were done on the same machine, and used Perron iteration for computing the maximum eigenvalue. The standard and periodic transfer matrices were compressed using the method of [7] thereby reducing the amount of RAM needed considerably. For the $N = 40$ 1-vertex transfer matrix the computation took 48 CPU-hours, used 0.5GB RAM for the matrix and also needed two vectors or the Perron iteration, each using 4GB RAM. For the C_{36} standard transfer matrix the computation took 988 CPU-hours, the automorphisms compressed transfer matrix used 203GB RAM and also needed two vectors, each using 20MB RAM.

Matrix	Time	RAM
$N = 40$ 1-vertex	48 CPU-hours	0.5GB matrix, two 4GB vectors
$N = 36$ periodic	988 CPU-hours	203GB matrix, two 20MB vectors

TABLE III
COMPUTATIONAL COSTS

We see that in order to reach a high prescision estimate of the asymptotic capacity the 1-vertex transfer matrix used less memory and less CPU time. The resources needed for the largest 1-vertex transfer matrix used here is today available on many larger workstations, while those needed for the largest periodic transfer matrix still requires a larger cluster. This means that the 1-vertex transfer matrix can first be used on a smaller inexpensive machine in order to find interesting examples of codes so that the more costly large computer will only be used when an interesting candidate has been found.

IV. THE GENERAL TWO DIMENSIONAL CASE

Let $\Gamma = (\langle k\rangle, E)$ be a directed graph, abbreviated as digraph, on the set of vertices $V = \langle k\rangle$, where $E \subset \langle k\rangle \times \langle k\rangle$. Let $X \subset \mathbb{Z}$. Then $\phi : X \to \langle k\rangle$ is viewed as a k colouring of X. ϕ is Γ-allowable colouring of X if $(\phi(i), \phi(i+1)) \in E$ whenever $i, i+1 \in X$. Denote by $\mathrm{C}_\Gamma(X)$ the set of allowable Γ-colourings of X. Note that a colour $j \in \langle k\rangle$ appears in $\phi \in \mathrm{C}_\Gamma(\mathbb{Z})$ if and only if the vertex j of Γ is contained in strongly connected component of Γ. Hence we assume that G is a disjoint union of strongly connected graphs. Recall that $\log\left[\mathrm{C}_\Gamma(\langle n\rangle)\right|, n = 1, 2, \ldots,$ is a subadditive sequence. Hence the $\mathbb{Z}$-entropy induced by Γ is given by

$$h_1(\Gamma) = \log\rho(\Gamma) = \lim_{n\to\infty} \frac{\log|\mathrm{C}_\Gamma(\langle n\rangle)|}{n}, \tag{IV.1}$$

where $\rho(\Gamma)$ is the spectral radius of Γ, i.e. the adjacency matrix corresponding to Γ. See for example [10], [11].

Let $\Gamma_1 = (\langle k\rangle, E_1), \Gamma_2 = (\langle k\rangle, E_2), E_1, E_2 \subseteq \langle k\rangle \times \langle k\rangle$ be two directed graphs on the set of vertices $V = \langle k\rangle$. We assume that each Γ_i is a disjoint union of strongly connected graphs. Let $X \subset \mathbb{Z}^2$. For each $i \in \mathbb{Z}$ denote

$$X_{1,i} = X \cap \mathbb{Z} \times \{i\}, \quad X_{2,i} = X \cap \{i\} \times \mathbb{Z}.$$

View $\phi : X \to \langle k\rangle$ as a k-colouring of X. Then ϕ is an allowable $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2)$-colouring of X if $\phi|X_{1,i}, \phi|X_{2,i}$ are Γ_1, Γ_2-allowable colourings respectively, for each $i \in \mathbb{Z}$. Let $\mathrm{C}_{\mathbf{\Gamma}}(X)$ be all $\mathbf{\Gamma}$-allowable colourings of X. It is known that the sequence $\log|\mathrm{C}_{\mathbf{\Gamma}}(\langle n_1\rangle \times \langle n_2\rangle)|$ is a subadditive sequence in one variable, where the other variable is fixed. Hence the $\mathbb{Z}^2$-entropy induced by $\mathbf{\Gamma}$ is given by, e.g. [10], [11],

$$h_2(\mathbf{\Gamma}) = \lim_{n_1,n_2\to\infty} \frac{\log|\mathrm{C}_{\mathbf{\Gamma}}(\langle n_1\rangle \times \langle n_2\rangle)|}{n_1 n_2} \le \frac{\log|\mathrm{C}_{\mathbf{\Gamma}}(\langle n_1\rangle \times \langle n_2\rangle)|}{n_1 n_2} \tag{IV.2}$$

for all $n_1, n_2 \in \mathbb{N}$.

Denote $\{i'\} = \{1, 2\}\backslash\{i\}$. Let $R_{n,i} = [r_{\phi,\psi,i}]_{\phi,\psi\in\mathrm{C}_{\Gamma_{i'}}(\langle n\rangle)}$ be the following $(0, 1)$ transfer matrices for $i = 1, 2$. $r_{\phi,\psi,i} = 1$ if the colouring of $\langle n\rangle \times \langle 2\rangle$, such that the colouring of $\{(1, 1), \ldots, (n, 1)\}$ and $\{(1, 2), \ldots, (n, 2)\}$ is given by ϕ and ψ respectively, is $(\Gamma_{i'}, \Gamma_i)$-allowable. Hence

$$\begin{aligned}
\log\rho(R_{n,1}) &= \lim_{n_1\to\infty} \frac{\log|\mathrm{C}_{\mathbf{\Gamma}}(\langle n_1\rangle \times \langle n\rangle)|}{n_1},\\
\log\rho(R_{n,2}) &= \lim_{n_2\to\infty} \frac{\log|\mathrm{C}_{\mathbf{\Gamma}}(\langle n\rangle \times \langle n_2\rangle)|}{n_2},\\
h_2(\mathbf{\Gamma}) &= \lim_{n\to\infty} \frac{\log\rho(R_{n,i})}{n} \le \frac{\log\rho(R_{n,i})}{n},\\
& i = 1, 2, \text{ for all } n \in \mathbb{N}.
\end{aligned} \tag{IV.3}$$

Note that for general Γ_1, Γ_2 we do not have lower bounds that converge to $h_2(\mathbf{\Gamma})$. In fact, there are examples that the computation of $h(\mathbf{\Gamma})$ is undecidable, e.g. [10], [11]. One has good lower bounds, similar to (I.2), if Γ_i is undirected for $i = 1$ or $i = 2$. Note that the isotropic case discussed in §2 is given by the condition that $\Gamma_1 = \Gamma_2$ is an undirected graph Δ. In that case $T_{n,2} = R_{n,1} = R_{n,2}$.

We now discuss the 1-vertex transfer matrix for a given $\mathbf{\Gamma}$. In this case we also have two transfer matrices $P_{n,1}, P_{n,2}$ which are $(0, 1)$ matrices. For simplicity of the exposition we discuss only $P_{n,2}$. Let $P_{n,2} = [p_{\phi,\psi,2}]_{\phi,\psi\in\mathrm{C}_{\Gamma_1}(\langle n\rangle)}$. Then $p_{\phi,\psi,2} = 1$ if and only if the conditions given in (I.5) hold, where $E = E_2$. The proof of Theorem 2.1 yields the inequality

$$\log\rho(P_{n,2}) \le \frac{\log\rho(R_{n,2})}{n} \text{ for } n \in \mathbb{N}. \tag{IV.4}$$

Hence

$$h_2(\mathbf{\Gamma}) \ge \limsup_{n\to\infty} \log\rho(P_{n,2}).$$

It is an interesting question when

$$h_2(\mathbf{\Gamma}) = \lim_{n\to\infty} \log\rho(P_{n,2}). \tag{IV.5}$$

Theorem 2.1 implies that the above equality holds if $\Gamma_1 = \Gamma_2$ is an undirected graph. We now give another condition for the equality (IV.5).

A colour $j \in \langle k\rangle$ is called a friendly colour, if $(j, i), (i, j) \in E_1 \cap E_2$ for all $i \in \langle k\rangle$. The existence of a friendly colour means that Γ_1 and Γ_2 are strongly connected graphs. Note that for the hard core model the colour 2 is friendly.

Theorem 4.1: Let $\Gamma_1 = (\langle k\rangle, E_1), \Gamma_2 = (\langle k\rangle, E_2)$ be two digraphs with a friendly colour. Then for each $n \geq 1$

$$\frac{\log \rho(R_{n,2})}{n+1} \leq \log \rho(P_{n+1,2}). \tag{IV.6}$$

Hence (IV.5) holds.

Proof: As in the proof of Theorem 2.1 we deduce that

$$|\mathrm{C}_{\mathbf{\Gamma}}(\langle n\rangle \times \langle m\rangle)| = \mathbf{1}^\top R_{n,2}^{m-1}\mathbf{1}.$$

Observe next that $\mathbf{1}^\top P_{n+1,2}^{(n+1)(m-1)}\mathbf{1}$ is the number of all $\omega \in \mathrm{C}_{\Gamma_1}(\langle (n+1)m\rangle)$ such that $(\omega(i), \omega(n+1+i)) \in E_2$ for $i = 1, \ldots, (m-1)(n+1)$. Let $\tau \in \mathrm{C}_{\mathbf{\Gamma}}(\langle n\rangle \times \langle m\rangle)$. Extend τ to $\hat\tau : \langle n+1\rangle \times m$ by letting $\hat\tau((n+1, i)) = j$ for $i = 1, \ldots, m$, where j is a friendly colour of $\mathbf{\Gamma}$. Then each such $\hat\tau$ is induced by $\omega \in \mathrm{C}_{\Gamma_1}(\langle (n+1)m\rangle)$ satisfying the above additional conditions. Hence

$$\mathbf{1}^\top R_{n,2}^{m-1}\mathbf{1} \leq \mathbf{1}^\top P_{n+1,2}^{(n+1)(m-1)}\mathbf{1}.$$

The arguments of the proof of Theorem 2.1 yield the inequality (IV.6). Combine this inequality with the inequality (IV.4) and the characterisation of $h_2(\mathbf{\Gamma})$ in (IV.3) to deduce (IV.5). □

V. The Three dimensional case

Let $\Gamma_i = (\langle k\rangle, E_i), E_i \subseteq \langle k\rangle \times \langle k\rangle$ be a digraph, which is a disjoint union of strongly connected graphs, for $i = 1, 2, 3$. Denote $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2, \Gamma_3)$. Let $\mathbf{e}_i = (\delta_{1i}, \delta_{i2}, \delta_{i3})^\top, i = 1, 2, 3$ be the standard basis in $\mathbb{Z}^3$. For $X \subset \mathbb{Z}^3$ the colouring $\phi : X \to \langle k\rangle$ is called $\mathbf{\Gamma}$-allowable if $(\phi(\mathbf{j}), \phi(\mathbf{j} + \mathbf{e}_i)) \in E_i$, whenever $\mathbf{j}, \mathbf{j} + \mathbf{e}_i \in X$, for $\mathbf{j} \in \mathbb{Z}^3$ and $i = 1, 2, 3$. Denote by $\mathrm{C}_{\mathbf{\Gamma}}(X)$ all $\mathbf{\Gamma}$-allowable colourings of X. Recall that the sequence $\log |\mathrm{C}_{\mathbf{\Gamma}}(\langle n_1\rangle \times \langle n_2\rangle \times \langle n_3\rangle)|$ is a subadditive sequence in one variable, where the other variables are fixed. Hence the $\mathbb{Z}^3$-entropy induced by $\mathbf{\Gamma}$ is given by, e.g. [10], [11],

$$\begin{aligned} h_3(\mathbf{\Gamma}) &= \lim_{n_1,n_2,n_3\to\infty} \frac{\log |\mathrm{C}_{\mathbf{\Gamma}}(\langle n_1\rangle \times \langle n_2\rangle \times \langle n_3\rangle)|}{n_1n_2n_3} \\ &\leq \frac{\log |\mathrm{C}_{\mathbf{\Gamma}}(\langle n_1\rangle \times \langle n_2\rangle \times \langle n_3\rangle)|}{n_1n_2n_3}, \end{aligned} \tag{V.1}$$

for all $n_1, n_2, n_3 \in \mathbb{N}$. Let $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$. Denote $\langle \mathbf{n}\rangle = \langle n_1\rangle \times \langle n_2\rangle$. As in the two dimensional case one can define the transfer matrices $R_{\mathbf{n},1}, R_{\mathbf{n},2}, R_{\mathbf{n},3}$ which are $(0, 1)$ matrices. For simplicity of the exposition we discuss only

$$R_{\mathbf{n},3} = [r_{\phi,\psi,3}]_{\phi,\psi\in \mathrm{C}_{(\Gamma_1,\Gamma_2)}(\langle \mathbf{n}\rangle)}.$$

$r_{\phi,\psi,3} = 1$ if the colouring of $\langle \mathbf{n}\rangle \times \langle 2\rangle$, such that the colouring of $\langle \mathbf{n}\rangle \times \{1\}, \langle \mathbf{n}\rangle \times \{2\}$ given by ϕ and ψ respectively, is $\mathbf{\Gamma}$-allowable. Hence

$$\begin{aligned} \log \rho(R_{(n_1,n_2),3}) &= \lim_{n_3\to\infty} \frac{\log |\mathrm{C}_{\mathbf{\Gamma}}(\langle n_1\rangle \times \langle n_2\rangle \times n_3)|}{n_3}, \\ h_3(\mathbf{\Gamma}) &= \lim_{n_1,n_2\to\infty} \frac{\log \rho(R_{(n_1,n_2),3})}{n_1n_2} \leq \frac{\log \rho(R_{(n_1,n_2),3})}{n_1n_2} \end{aligned} \tag{V.2}$$

for all $n_1, n_2 \in \mathbb{N}$.

We now discuss the 1-vertex transfer matrix for a given $\mathbf{\Gamma}$. In this case we have three transfer matrices $P_{\mathbf{n},1}, P_{\mathbf{n},2}, P_{\mathbf{n},3}$ which are $(0, 1)$ matrices. For simplicity of the exposition we discuss only $P_{\mathbf{n},3}$. Let $\hat{\mathrm{C}}_{\Gamma_1}(\langle n_1n_2\rangle)$ be the subset of all Γ_1-allowable colourings of $\phi \in \mathrm{C}_{\Gamma_1}(\langle n_1n_2\rangle)$ satisfying the additional condition:

$$(\phi(i), \phi(n_1 + i)) \in E_2 \text{ for } i = 1, \ldots, n_1(n_2 - 1). \tag{V.3}$$

The condition (V.3) yields that the colouring given by ϕ on $\langle n_1n_2\rangle$ induces a (Γ_1, Γ_2)-allowable colouring of $\langle \mathbf{n}\rangle \times \{1\}$ as discussed in the previous section.

Let $P_{\mathbf{n},3} = [p_{\phi,\psi,3}]_{\phi,\psi\in \hat{\mathrm{C}}_{\Gamma_1}(\langle n_1n_2\rangle)}$. Then $p_{\phi,\psi,3} = 1$ if and only if $\psi(i) = \phi(i + 1)$ for $i = 1, \ldots, n_1n_2 - 1$ and

$$(\phi(1), \psi(n_1n_2)) \in E_3. \tag{V.4}$$

The condition (V.4) yields that if we colour $\langle \mathbf{n}\rangle \times 1$ in ϕ and $(1, 1, 2)$ in the colour $\psi(n_1n_2)$, we obtain a $\mathbf{\Gamma}$-allowable colouring of $\langle \mathbf{n}\rangle \times \{1\} \cup \{(1, 1, 2)\}$.

Observe that $\mathbf{1}^\top P_{\mathbf{n},3}^{n_1n_2(m-1)}\mathbf{1}$ consists of all $\omega \in \mathrm{C}_{\Gamma_1}(n_1n_2m)$ satisfying the above "periodic" conditions

1) $(\omega(i), \omega(i + n_1)) \in E_2,\ i = 1, \ldots, n_1(n_2m - 1)$.
2) $(\omega(i), \omega(i + n_1n_2) \in E_3,\ i = 1, \ldots, n_1n_2(m - 1)$.

Hence each such ω induces a $\mathbf{\Gamma}$-allowable colouring of $\langle \mathbf{n}\rangle \times \langle m\rangle$. Therefore

$$\mathbf{1}^\top P_{\mathbf{n},3}^{n_1n_2(m-1)}\mathbf{1} \leq \mathbf{1}^\top R_{\mathbf{n},3}^{m-1}\mathbf{1}.$$

The proof of Theorem 2.1 yields the inequality

$$\log \rho(P_{(n_1,n_2),3}) \leq \frac{\log \rho(R_{(n_1,n_2),3})}{n_1n_2} \text{ for } n_1, n_2 \in \mathbb{N}. \tag{V.5}$$

Hence

$$h_3(\mathbf{\Gamma}) \geq \limsup_{n_1,n_2\to\infty} \log \rho(P_{\mathbf{n},3}).$$

It is an interesting question when

$$h_3(\mathbf{\Gamma}) = \lim_{n_1,n_2\to\infty} \log \rho(P_{(n_1,n_2),3}). \tag{V.6}$$

A colour $j \in \langle k\rangle$ is called a friendly colour, if $(j, i), (i, j) \in E_1 \cap E_2 \cap E_3$ for all $i \in \langle k\rangle$.

Theorem 5.1: Let $\Gamma_1 = (\langle k\rangle, E_1), \Gamma_2 = (\langle k\rangle, E_2), \Gamma_3$ be three digraphs with a friendly colour. Then for each $n_1, n_2 \geq 1$

$$\frac{\log \rho(R_{(n_1,n_2),3})}{(n_1 + 1)(n_2 + 1)} \leq \log \rho(P_{(n+1,n_2+1),3}). \tag{V.7}$$

Hence (V.6) holds.

Proof: Let $\phi \in \mathrm{C}_{\mathbf{\Gamma}}(\langle n_1\rangle \times \langle n_2\rangle \times \langle m\rangle)$. Extend ϕ to the colouring $\hat\phi : \langle n_1 + 1\rangle \times \langle n_2 + 1\rangle \times \langle m\rangle$ by colouring the additional points in a friendly colour. Note that $\hat\phi$ induces the colouring $\omega \in \mathrm{C}_{\Gamma_1}((n_1 + 1)(n_2 + 1)m)$ such that

$$\begin{aligned} &(\omega(i), \omega(i + n_1 + 1)) \in E_2, \\ &i = 1, \ldots, (n_1 + 1)(n_2 + 1)m - n_1 - 1, \\ &(\omega(j), \omega(j + (n_1 + 1)(n_2 + 1))) \in E_3, \\ &j = 1, \ldots (n_1 + 1)(n_2 + 1)(m - 1). \end{aligned} \tag{V.8}$$

Hence

$$\mathbf{1}^\top R_{(n_1,n_2),3}^{m-1}\mathbf{1} \leq \mathbf{1}^\top P_{(n_1+1,n_2+1),3}^{(n_1+1)(n_2+1)(m-1)}\mathbf{1}.$$

Therefore (V.7) holds. Use (V.5) and (V.2) to deduce (V.6). □

We now discuss briefly the isotropic symmetric case, i.e. $\Gamma_1 = \Gamma_2 = \Gamma_3 = \Delta = (\langle k\rangle, E)$ is an undirected graph with no isolated vertices. Then the transfer matrix $T_{(n_1,n_2),3}$ is equal to $R_{(n_1,n_2),3}$. Denote by $T_{(n_1,n_2),3,\mathrm{per}}$ the transfer matrix corresponding to all (n_1, n_2) periodic colourings, which are Δ-allowable colourings of $(\mathbb{Z}/n_1) \times (\mathbb{Z}/n_2)$. Note that $T_{(n_1,n_2),3,\mathrm{per}}$ is a principal submatrix of $T_{(n_1,n_2),3}$. Recall the following improvement of (V.2) [10], [11], [5]

$$h_3(\mathbf{\Gamma}) = \lim_{n_1,n_2\to\infty} \frac{\log\rho(T_{(2n_1,2n_2),3,\mathrm{per}})}{4n_1n_2} \le \frac{\log\rho(T_{(2n_1,2n_2),3,\mathrm{per}})}{4n_1n_2}, \quad \text{(V.9)}$$

for all $n_1, n_2 \in \mathbb{N}$.

Let $S_{(n_1,n_2),3} = P_{(n_1,n_2),3}$ be the 1-vertex transfer matrix corresponding to Δ. A given Δ-allowable colouring on $\mathbb{Z} \times \mathbb{Z} \times \langle m\rangle$, which is (n_1, n_2) periodic on each $\mathbb{Z} \times \mathbb{Z} \times \{i\}$ for each $i \in \langle m\rangle$, induces a colouring $\phi \in \mathrm{C}_\Delta(\langle n_1 + 1\rangle \times \langle n_2 + 1\rangle \times \langle m\rangle)$. This colouring corresponds to the colouring $\omega \in \mathrm{C}_\Delta((n_1+1)(n_2+1)m)$ that satisfies the conditions (V.8), where $E_2 = E_3 = E$. Hence we have an analogue of (V.7)

$$\frac{\log\rho(T_{(n_1,n_2),3,\mathrm{per}})}{(n_1 + 1)(n_2 + 1)} \le \log\rho(S_{(n_1+1,n_2+1),3}). \quad \text{(V.10)}$$

Use (V.9), (V.5) and (V.2) to deduce

$$\lim_{n_1,n_2\to\infty} \log\rho(S_{(2n_1+1,2n_2+1),3}) = h_3(\Delta). \quad \text{(V.11)}$$

VI. The Three-dimensional Run Length Limited Channel

The capacity of the three-dimensional channel corresponding to the isotropic symmetric case of $\mathbb{Z}^3$ with the same digraph as in Section III is the three-dimensional $(0, 1)$ run length limited channel. This channel was considered in [6] where the transfer matrix methods previously used for the two-dimensional case was extended to find the bounds

$$1.43644 \le e^{h_3(\mathbf{\Gamma})} \le 1.44082$$

This case has also been studied in the physics literature, see e.g. [13], in the form of a three-dimensional lattice gas, where colourings are weighted according to their number of 1s. Mathematical results on the structure of the set of colourings have shown [14], [15] that many of the ordinary Monte Carlo algorithms used to study this model numerically have slow mixing properties and that there are interesting long range correlations in the positions of the 1s.

A. Computational results

We first used the standard transfer matrix method in combination with the compression method of [7] to compute the largest eigenvalue for both the periodic and aperiodic graphs needed to bound the capacity for this channel. In order to verify our program we recomputed the eigenvalues from [6] and the results agreed. In Table IV we show the eigenvalues not previously computed. Using these eigenvalues as in [1], [5] we find the following bounds for the exponentiated capacity:

$$1.4365871627266 \le e^{h_3(\mathbf{\Gamma})} \le 1.43781634614 \quad \text{(VI.1)}$$

We have here determined one more decimal in the value of $e^{h_3(\mathbf{\Gamma})}$, and limited the next decimal to only two possible values.

The largest eigenvalues for the 1-vertex transfer are given in Table V. As we can see the eigenvalues give a reasonably good approximation of the capacity. Even for a small case like (4,4) the first three decimals agree with the bounds in VI.1. For the largest case the eigenvalue is in the interval given in VI.1, but here we do not have an error bound. At first sight the values given by the 1-vertex transfer matrix looks less impressive for this case, but in fact the best bounds before the current paper only gave $e^{h_3(\mathbf{\Gamma})} = 1.4$. Hence even the $(4, 4)$ case for the 1-vertex transfer matrix gave an estimate with improved accuracy.

n_1	n_2	
4	4	1.431707
4	5	1.433880
5	4	1.433943
5	5	1.439764
6	5	1.436801

TABLE V
LARGEST EIGENVALUE FOR THE 1-VERTEX TRANSFER MATRIX

B. Computational costs

The most striking feature for the three-dimensional case is the difference in computational resources needed for the two approaches.

The computation of the eigenvalues for $C_{6,8}$ used 3700 GB RAM and ran for 10.6 hours on 512 4-core CPUs on a linux cluster, giving a total of about 21700 CPU-hours. The program was a Fortan 90 program using OpenMP and MPI for communication. Before compression the matrix had side 1682382.

For the 1-vertex transfer matrix constraints, in programming time and computer access, made us settle for a single machine implementation of the method in Mathematica which was run on a desktop Macintosh with 2GB RAM. For $n_1 = 6, n_2 = 5$ the computation used 52 CPU-hours. The matrix has side 339 000 but since it has only two non-zero positions in each row the amount of RAM used was negligible.

Matrix	Time	RAM
$n_1 = 6, n_2 = 5$ 1-vertex	52 CPU-hours	5MB matrix
$C_6 \times C_8$ periodic	21700 CPU-hours	3700GB matrix

TABLE VI
COMPUTATIONAL COSTS FOR THREE-DIMENSIONS

VII. The monomer-dimer model

The classical monomer-dimer model on $\mathbb{Z}^d$ consists of tiling $\mathbb{Z}^d$ with monomers and dimers in the direction $\mathbf{e}_i = (\delta_{1i}, \ldots, \delta_{di})^\top$ for $i = 1, \ldots, d$. This tiling can be coded in

n_1	n_2	aperiodic	periodic	aperiodic/periodic
3	12			769999.6006127802
4	7	41543.31662520356		
4	8	184654.5439467464	117151.9963311473	
4	10		215347.9226316121	
5	5	13427.06985344107	8185.111027254276	10331.06553679985
5	6	85738.84889761954	51133.96879131764	68793.17993810770
5	7	547489.0319884565	307078.2654460451	
5	8			2779905.231816480
6	6	786528.5060953929	482862.3074476483	
6	8		37133338.84386827	

TABLE IV
LARGEST EIGENVALUE FOR THE STANDARD TRANSFER MATRIX

$2d+1$ colours, see [11], [5]. That is, the colour $2d+1$ corresponds to a monomer, and the colours $2i-1, 2i$ correspond a dimer in the direction $\mathbf{e}_i$. So $\Gamma_i = (\langle 2d+1\rangle, E_i)$ is given by the following conditions on E_i for $i = 1, \ldots, d$.

- All vertices $\langle 2d+1\rangle\backslash\{2i-1, 2i\}$ form a complete directed graph on $2d-1$ vertices.
- $(j, 2i-1), (2i, j) \in E_i$ for all $j \in \langle 2d+1\rangle\backslash\{2i-1, 2i\}$.
- $(2i-1, 2i) \in E_i$.

Denote by h_d the monomer-dimer entropy, e.g. [11], [5]. Let $\mathbf{n} = (n_1, \ldots, n_{d-1}) \in \mathbb{N}^{d-1}$. Define the 1-vertex transfer matrix $P_{\mathbf{n},d}$ as in the previous sections for $d = 2, 3$. We claim that

$$h_d = \lim_{n_1,\ldots,n_{d-1}\to\infty} \log \rho(P_{(n_1,\ldots,n_{d-1}),d}). \qquad \text{(VII.1)}$$

In this case the colour $2d+1$ is not friendly. It is shown in [5] that to compute the d-dimer entropy h_d it is enough to consider the tiling of the box ϕ of $\langle n_1\rangle \times \ldots \times \langle n_d\rangle$ with monomers and dimers that fit inside the box. In this case we can extend the tiling ϕ to the tiling $\tilde{\phi}$ of $\langle n_1+1\rangle \times \ldots \times \langle n_{d-1}+1\rangle \times \langle n_d\rangle$ by adding an additional layer of monomers. The arguments of the previous section yield the equality (VII.1).

ACKNOWLEDGEMENTS

This research was conducted using the resources of High Performance Computing Center North (HPC2N).

REFERENCES

[1] Z. Nagy and K. Zeger, "Capacity bounds for the 3-dimensional $(0, 1)$ runlength limited channel," in *Applied algebra, algebraic algorithms and error-correcting codes (Honolulu, HI, 1999)*, ser. Lecture Notes in Comput. Sci. Berlin: Springer, 1999, vol. 1719, pp. 245–251.

[2] N. J. Calkin and H. S. Wilf, "The number of independent sets in a grid graph," *SIAM J. Discrete Math.*, vol. 11, no. 1, pp. 54–60 (electronic), 1998.

[3] R. J. Baxter, I. G. Enting, and S. K. Tsang, "Hard-square lattice gas," *J. Statist. Phys.*, vol. 22, no. 4, pp. 465–489, 1980.

[4] A. Kato and K. Zeger, "On the capacity of two-dimensional run-length constrained channels," *IEEE Trans. Inform. Theory*, vol. 45, no. 5, pp. 1527–1540, 1999.

[5] S. Friedland and U. N. Peled, "Theory of computation of multidimensional entropy with an application to the monomer-dimer problem," *Adv. in Appl. Math.*, vol. 34, no. 3, pp. 486–522, 2005.

[6] Z. Nagy and K. Zeger, "Asymptotic capacity of two-dimensional channels with checkerboard constraints," *IEEE Trans. Inform. Theory*, vol. 49, no. 9, pp. 2115–2125, 2003.

[7] P. H. Lundow and K. Markström, "Exact and approximate compression of transfer matrices for graph homomorphisms," *LMS J. Comput. Math.*, vol. 11, pp. 1–14, 2008.

[8] C. Borgs, J. Chayes, L. Lovász, V. T. Sós, and K. Vesztergombi, "Counting graph homomorphisms," in *Topics in discrete mathematics*, ser. Algorithms Combin. Berlin: Springer, 2006, vol. 26, pp. 315–371.

[9] M. Freedman, L. Lovász, and A. Schrijver, "Reflection positivity, rank connectivity, and homomorphism of graphs," *J. Amer. Math. Soc.*, vol. 20, no. 1, pp. 37–51 (electronic), 2007.

[10] S. Friedland, "On the entropy of $\mathbf{Z}^d$ subshifts of finite type," *Linear Algebra Appl.*, vol. 252, pp. 199–220, 1997.

[11] ——, "Multi-dimensional capacity, pressure and Hausdorff dimension," in *Mathematical systems theory in biology, communications, computation, and finance (Notre Dame, IN, 2002)*, ser. IMA Vol. Math. Appl. New York: Springer, 2003, vol. 134, pp. 183–222.

[12] E. H. Lieb, "Residual Entropy of Square Ice," *Physical Review*, vol. 162, pp. 162–172, Oct. 1967.

[13] H. C. Marques Fernandes, Y. Levin, and J. J. Arenzon, "Equation of state for hard-square lattice gases," *Physical Review E*, vol. 75, no. 5, pp. 052 101–+, May 2007.

[14] D. Galvin, "Sampling independent sets in the discrete torus," *Random Structures Algorithms*, vol. 33, no. 3, pp. 356–376, 2008.

[15] D. Galvin and J. Kahn, "On phase transition in the hard-core model on $\mathbf{Z}^d$," *Combin. Probab. Comput.*, vol. 13, no. 2, pp. 137–164, 2004.