Factors of r-partite graphs and bounds for the Strong Chromatic Number

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Abstract

We give an optimal degree condition for a tripartite graph to have a spanning subgraph consisting of complete graphs of order 3. This result is used to give an upper bound of 2Δ for the strong chromatic number of n vertex graphs with $\Delta \geq n/6$.

1 Introduction

The basic graph theoretical terms and notation not defined here can be found in [Die97]. A *balanced r-partite graph* is a an r-partite graph with vertex set V partitioned into $V_0 \cup \ldots \cup V_{r-1}$ such that $|V_i| = n$. We may think of such a graph as a subgraph of $K_r(n)$, the *blow-up* of K_r, where the blow-up $G(n)$ of a general graph G is obtained by replacing each vertex v_i in G with an independent set V_i of size n and each edge v_iv_j with a complete bipartite graph $K(V_i, V_j)$.

We say that a graph G has a *K_r-factor* if it contains n vertex disjoint r-cliques — copies of the complete graph on r vertices. Hence a K_r-factor is a spanning subgraph with components which are complete graphs on r vertices. In this paper we prove the following theorem.

Theorem 1. *If $G \subset K_3(n)$ with $\delta(G) \geq \frac{3}{2}n$ then G has a K_3-factor.*

This theorem is optimal for a minimum degree condition. To see that, join the vertices in V_0 to all vertices of $V_1 \cup V_2$ and then join the vertices of V_1 to V_2 such that the bipartite graph between V_1 and V_2 contains no perfect matching but has minimum degree at least $\frac{1}{2}n - 1$.

For general K_r-factors, we obtain the following result.

Theorem 2. *Let $l_s := \sum_{k=1}^{s} 1/k$. Any subgraph G of $K_r(n)$ with minimum degree*

$$\delta(G) > (r - 1 - 1/(1 + l_{r-2}))n + (r - 1)l_{r-2}/(1 + l_{r-2}),$$

has a K_r-factor.

This is based on a minimum degree condition, (2) below, by G. Jin for finding an r-clique. However, this is far from optimal for large r as is demonstrated by Alon's and Haxell's result in [Alo92] and [Hax04] discussed below.

The problem finding minimum degree conditions for finding K_r-factors in balanced r-partite graphs can also been formulated as the problem finding a maximum degree condition for the *strong chromatic number* of graphs. If $G = (V, E)$ is a graph then the strong chromatic number of G, denoted $s_\chi(G)$, is the minimum n such that the following hold: Any graph being the union of G and a set of vertex-disjoint n-cliques is n-colourable. Here we take the union of edges, adding vertices to G if necessary. Taking the complementary graph, we see that this is exactly asking for a K_r-factor in a balanced r-partite graph, each part having n vertices. Note that, for the complete bipartite graph $K_{n,n}$, we get $s_\chi(K_{n,n}) = 2\Delta(K_{n,n})$.

In [Alo92] N. Alon proves that $s_\chi(G) \leq K\Delta(G)$ for a quite large constant K. The value obtained is $K = 2^{20000}$. In [MR02] it is pointed out that a careful calculation would reduce it to $K = 10^{10}$ the constant is by several authors believed to be smaller.

Conjecture 1. *For all graphs G,*

$$s_\chi(G) \leq 2\Delta(G)$$

The best bound published so far is $s_\chi(G) \leq 3\Delta(G) - 1$, given by Haxell in [Hax04]. From Theorem 1, we conclude that the strong chromatic number can be bounded by $2\Delta(G)$ if $|V(G)| \leq 6\Delta(G)$. This result should not be compared with the more complete results of Alon and Haxell. But, the authors think that the tripartite case covered above has its own interest, apart from verifying the conjectured bound for this case.

Actually, the problem of finding degree conditions that guarantees an r-clique in vertex balanced r-partite graphs seems to be far from settled. Following notation in [Bol78], we take $\delta_r(n)$ as the largest minimum degree of a K_r-free subgraph of $K_r(n)$. In [Jin92], G. Jin proves that $\delta_4(G) =$

$\lceil(2+\frac{1}{3})n\rceil$ and it is proved to be a sharp minimum degree condition. In fact it is proved that

$$\lim_{r\to\infty}(r-\delta_r(n)/n) \geq \frac{3}{2}. \tag{1}$$

In particular, this means that the proof of Theorem 1 cannot be generalised immediately to the r-partite case. For general $r \geq 3$, Jin obtains the upper bound

$$\delta_r(n) \leq (r-1-1/l_{r-2})n, \tag{2}$$

where $l_s := \sum_{k=1}^{s} 1/k$ and this is essentially tight for $r = 3, 4$. For $r = 3$, the bound $\delta_3(n) \leq n$ is a result by Graver (see [Bol78]) which is used in the proof of Theorem 1.

In [Bol78] it is conjectured that the inequality in (1) is actually an equality. Note that the result by Alon on the strong chromatic number — or equivalently a K_r-factor — gives a nontrivial upper bound for $\sup_n \delta_r(n)/n$. Such a bound is posed as an open problem in this, admittedly dated, reference book.

Another way of developing this question would be to view Theorem 1 as a condition implying that a subgraph of the graph $C_3(n)$ has a C_3-factor. Here, $C_r(n)$ denotes the blow-up of the r-cycle $C_r = \{v_i v_{i+1} : i = 0, \ldots, r\}$ with indices reduced modulo r. A natural question would be to determine minimum degree conditions for a cyclic C_r-factor in a subgraph G of a $C_r(n)$, where a "cyclic C_r-factor" means that each of the n components in the factor is an r-cycle containing exactly one vertex from each of the blown up vertices V_i, $i = 0, \ldots, r-1$. Thus, a C_r-factor need not be cyclic for even r.

A result similar to that of Theorem 1 for cycle factors is the following for which we supply a sketch of proof.

Theorem 3. *If $G \subset C_r(n)$ with $\delta(G) \geq \frac{3}{2}n+2$ then G has a cyclic C_r-factor.*

A more refined, and longer, version of our proof will bring the degree condtion down to $\frac{3}{2}n+1$. A construction similar to that following Theorem 1 shows that $3n/2$ is a lower bound on the degrees to ensure that C_r-factor exists in $C_r(n)$. We conjecture that this is in fact the correct bound.

Conjecture 2. *If $G \subset C_r(n)$ with $\delta(G) \geq \frac{3}{2}n$ then G has a cyclic C_r-factor.*

2 Proofs and remarks

Let G be a balanced tripartite graph satisfying the conditions of Theorem 1. Induced subgraphs are denoted by $G[S]$, where $S \subset V(G)$. For $S \subset V(G)$, we use the notation $d(x, S)$ for the number of edges in G joining the vertex x with vertices in the set $S \subset V(G)$. For a subgraph H, let $d(x, H)$ means $d(x, V(H))$. When we take the cardinality of a graph, as in $|F|$, we mean the number of edges in F. Let the three parts of G be denoted V_0, V_1, V_2 and when refering to one of these parts, say V_i, the index i, should implicitly be reduced modulo three so that the two other parts can be referred to as V_{i+1} and V_{i-1}, say.

2.1 Proof of Theorem 1

We assume to arrive at a contradiction that

$$G \text{ admits no full } K_3\text{-factor.} \tag{3}$$

We assume moreover that G is an edge maximal counterexample, so that we get a K_3-factor by adding any edge to G. Thus G has plenty of *configurations*, by which we mean an incomplete K_3-factor $F = F_1 \cup \cdots \cup F_{n-1}$ of $n-1$ vertex disjoint copies of K_3 in G. Denote by $X = X(F) = V(G) \setminus V(F)$, the three vertices not contained in F, where x_i denotes the element of X in part V_i, for $i = 0, 1, 2$. For a vertex u in $V \setminus X$, let F_u denote the unique triangle in F that covers u.

A vertex $u \in V_i \setminus \{x_i\}$ is *exchangeable* relative the current configuration if x_i makes up a triangle $T_u = G[\{x_i\} \cup V(F_u) \setminus \{u\}]$ together with the other vertices of the clique F_u in F containing u, i.e. if $d(x_i, F_u) = 2$. Let $Y = Y(F)$ denote the set of exchangeable vertices and let $Y_i = Y \cap V_i$. Since $d(v) \geq 3n/2$, we have at least

$$|Y_i| \geq d(x_i, V \setminus X) - (n-1) \geq n/2 + 1 - d(x_i, X) \tag{4}$$

exchangeable vertices in the part V_i.

If $u \in V_i$ is exchangeable, we may *exchange* or *interchange* u with x_i in the obvious manner: We obtain the new configuration $F' = (F \setminus F_u) \cup T_u$. Note that, after this exchange, x_i will be an exhangeable vertex in F'. After this operation, the set of exchangeable vertices, $Y' = Y(F')$, relative F' will coincide with the set of exchangeable vertices $Y = Y(F)$ relative F except, possibly, in the part V_i and on the vertices of $V(F_u)$, i.e. $Y(F) \triangle Y(F') \subset V_i \cup V(F_u)$. It follows that a subset $S \subset X \cup Y$ of at most three exchangeable vertices, such that

$$|S \cap V_i| \leq 1$$

and such that for all components F_j of F

$$|S \cap F_j| \leq 1$$

is *free* in the following sense: We can exchange the vertices in $S \setminus X$ one by one to obtain a configuration F' such that $S \subset X' = V(G) \setminus V(F')$.

From (3) we deduce that

$$H \text{ contains no free triangle } T, \tag{5}$$

i.e. a subgraph $T \cong K_3$ such that $V(T)$ is a free set, since exchanging $V(T)$ for X would give a full K_3-factor.

Let $H = H(F) = G[X \cup Y]$ denote the subgraph of G induced on the set of exchangeable vertices and X. We will consider the following properties of the configuration F

$$G[X] \text{ contains zero edges} \tag{X0}$$

$$G[X] \text{ contains one edge,} \tag{X1}$$

$$G[X] \text{ contains two edges.} \tag{X2}$$

$$H = G[X \cup Y] \text{ contains a triangle,} \tag{T}$$

We can clearly exclude the case that $G[X]$ has three edges, since that would mean that $F \cup G[X]$ is a full K_3-factor.

Let (A) $\rightsquigarrow$ (B) mean the following: Given a configuration F satisfying the property (A), we can either reach a contradiction to our assumption (3) that G contains no K_3-factor or, by a series of legal exchanges, reach a configuration F' that satisfies the property (B). We say that property (A) can be *reduced* to property (B). The theorem is proved as soon as we prove the following two lemmas. The first lemma allow us to reduce to the case (X0).

Lemma 4. *We have the following reductions.*

1. (T) $\rightsquigarrow$ (X0).
2. (X1) $\rightsquigarrow$ (T).
3. (X2) $\rightsquigarrow$ (X0) $\vee$ (X1) $\vee$ (T).

The following lemma takes care of the remaining case.

Lemma 5. *The property* (X0) *implies that G contains a full K_3-factor and thus leads to a contradiction with* (3).

2.2 Proof of Lemma 4

Proof of (T) $\rightsquigarrow$ (X0). Let T be a triangle of H. As pointed out above, we can exlude the case that T is free and hence T must share at least one edge with F.

We reduce first to the case when T is contained in F: Assume without loss of generality that, $T = u_0w_1u_2$, say, where $u_i \in Y_i$ and where $u_0u_1u_2 = F_{u_1}$ is triangle in F. If $w_1 \neq u_1$, we obtain the case with one of the triangles of H is entirely contained in F by, if $x_1 \neq w_1$, interchanging x_1 with w_1. (If $x_1 = w_1$ we need to do nothing.) In this new configuration F', the vertex u_1 is exchangeable, together with u_0 and u_2 and thus $F_{u_1} = u_0u_1u_2$ is a triangle in $F' \cap H'$. The situation is depicted in the right hand side of Figure 1.

Thus we have reduced to the case when $T \subset H \cap F$. As is demonstrated in figure 1, this implies that $G[X] = \emptyset$: If, say, the edge x_0x_2 was present, then $x_0u_1x_2$ is a free triangle; the edges x_0u_1 and u_1x_2 are due to the fact that both u_0 and u_1 are exchangeable vertices. $\square$

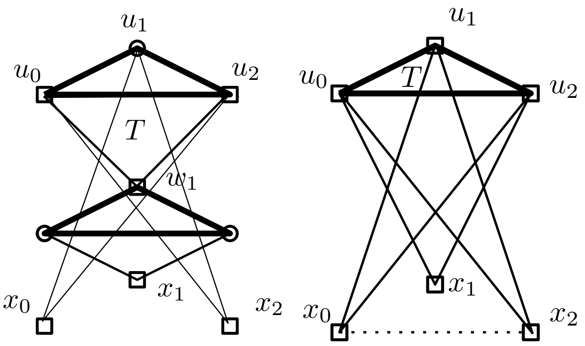


Figure 1: Left: The case when one edge of the triangle T belongs to F. Right: The case when $T \subset F \cap H$. Fat edges are edges in F and square vertices are vertices of $X \cup Y$.

Proof that (X1) $\rightsquigarrow$ (T). If $G[X]$ contains exactly one edge. Then $d(x_i, X) \leq 1$, for $i = 0, 1, 2$ and we obtain, on account of (4), that

$$|Y_i| \geq n/2, \quad \text{for } i = 0, 1, 2. \tag{6}$$

We show that

$$(6) \implies (\mathrm{T}). \tag{7}$$

By (6) we have $|(X \cup Y) \cap V_i| \geq n/2 + 1$ and we can take a balanced induced subgraph H' of $H = G[Y \cup X]$ with $n' = \lceil n/2 \rceil + 1$ vertices in each part. For $u' \in H'$, we have degree

$$d(u, H') \geq d(u) - d(u, V \setminus V(H')) \geq 3n/2 - 2n + 2n' > n'.$$

By Gravers bound, i.e. the bound (2) for $r = 3$, $d(u, H') > \delta_3(n') = n'$ and hence H' contains a triangle, which means that we have reduced to the case (T). $\square$

Proof that (X2) $\rightsquigarrow$ (X0) $\vee$ (X1) $\vee$ (T). Assume without loss of generality that $G[X] = \{x_0x_1, x_1x_2\}$. By (4), $|Y_i| \geq n/2$ for $i = 0, 2$ and $|Y_1| \geq n/2 - 1$. We may assume that $|Y_1| = n/2 - 1$ since we would otherwise have (6) which by (7) implies (T). That (4) holds with equality for $|Y_1|$ implies that for all triangles F_j in the partial factor F, we have

$$d(x_1, F_j) \geq 1, \tag{8}$$

since otherwise the counting in (4) gives a higher number.

If $i = 0$ let $\bar{i} = 2$ and if $i = 2$ let $\bar{i} = 0$. We have at least

$$d(x_i) - d(x_i, X) - d(x_i, V \setminus (Y \cup X)) \geq 3n/2 - 1 + 2(n-1) + |Y_1| + |Y_{\bar{i}}|$$

edges between x_i and $Y \cup X$. Since $|Y_{\bar{i}}| \geq n/2$, we get $d(x_i, H) \geq |Y_1| + 1$ which implies that $d(x_i, Y_{\bar{i}}) \geq 1$. It follows there is a pair $(z_0, z_2) \in Y_0 \times Y_2$ such that x_i is adjacent to $z_{\bar{i}}$, for $i = 0, 2$. Note that, neither z_0 nor z_2 can be adjacent to x_1, since each edge would give rise to a free triangle $x_0z_2x_1$ (or $x_2z_0x_1$). By (8) this means that z_0 and z_2 cannot belong to the same triangle F_j of F and therefore $\{z_0, z_2\}$ is a free set. By exchanging $\{x_0, x_2\}$ with $\{z_0, z_2\}$ we obtain a configuration such that $G[X'] \subset \{z_0z_2\}$ and thus reduce to the case (X0) or (X1). $\square$

2.3 Proof of lemma 5

By (4), we have $|Y_i| \geq n/2 + 1$ and thus

$$|(X \cup Y) \cap V_i| \geq n' = \lceil n/2 \rceil + 2.$$

Let H' be a balanced induced subgraph of $G[X \cup Y]$ on exactly $3n'$ vertices. Then,

$$d(x, H') \geq \lceil 3n/2 \rceil - (2n - 2n') \geq \lceil n/2 \rceil + 4 = n' + 2. \tag{9}$$

We orient the edges of H' so that the edge uv is oriented $\overrightarrow{uv}$ if $u \in V_i$ and $v \in V_{i+1}$. For $x \in V(H')$, let $d^+(x)$ and $d^-(x)$ denote the out-degrees and in-degrees in this orientation of H', respectively.

Assume that $\overrightarrow{uv} \in H'$ is a free edge, i.e. $F_u \neq F_v$. Since H' is balanced we know that

$$|N(u, H') \cap N(v, H')| \geq d^-(u) + d^+(v) - n'. \tag{10}$$

If it holds that

$$d^-(u) \geq d^-(v) \tag{11}$$

then, since $d^-(u) + d^+(v) = d^-(u) + d(v, H') - d^-(v)$, we get from (10) and (9) that

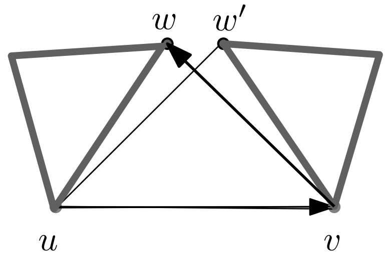
$$|N(u, H') \cap N(v, H')| \geq d(v, H') - n' \geq 2. \tag{12}$$

Thus, (11) implies that the edge uv is contained in *at least two* triangles $T = uvw$ and $T' = uvw'$ contained in H', with $w \neq w'$. Since we assumed that uv is a free edge, both triangles must contain exactly one edge from F, i.e., say, $F_u = F_w$ and $F_v = F_{w'}$, since otherwise we have obtained an free triangle. We cannot have the case that $F_u = F_v$ since we assumed that uv was free and it also follows that vw is a free edge. Note also that this means the following: For any free edge $\overrightarrow{uv} \in H'$ satisfying (11), there is a

$$\text{continuing free edge } \overrightarrow{vw} \text{ such that } uw \in F. \tag{13}$$

The situation is illustrated in figure 2.3.

Figure 2: Condition (11) yields two triangles containing uv. Each must share an edge with a triangle in F, and thus a free edge $\vec{vw}$ that continues $\vec{uv}$, such that $uw \in F$.



Moreover, if the inequality (11) is strict then $|N(u) \cap N(v)| \geq 3$ and we obtain a third, then a necessarily free triangle. It follows that $d^-(u) \leq d^-(v)$ for all free edges $\overrightarrow{uv} \in H'$. In other words d^- is nondecreasing in the forward direction along free edges.

Let S be the set

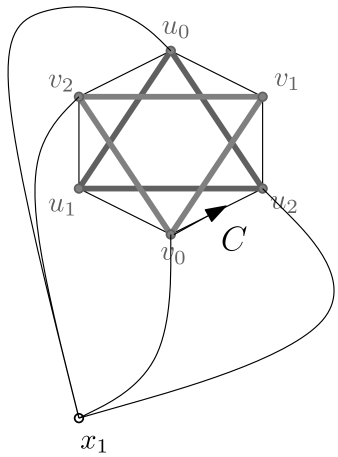
$$S = \{u \in V(H') : d^-(u) = \max_{v \in V(H')} d^-(v)\} \tag{14}$$

of vertices of maximum in-degree d^-. This is therefore an absorbing set for the oriented graph H'. Here, (11) is satisfied with equality along all edges $\overrightarrow{uv}$, $uv \in H'[S]$. Since

$$d^+(u) = d(u, H') - d^-(u) \geq 2,$$

each vertex $u \in S$ has at least one forward free edge $\overrightarrow{uv}$, where the endpoint v necessarily belongs to S.

Figure 3: The oriented 6-cycle $\vec{C}$ with inscribed triangles from F. We must have two free triangles containing x_1, since u_1 and v_1 are both exchangeable.



Moreover, by (13), given a free forward edge $\overrightarrow{uv}$ in $H'[S]$, we there is a continuation $\overrightarrow{vw}$, such that vw is free and $uw \in F$. We must have $w \in S$, since in-degree d^- is non-decreasing and hence we can repeat this construction. Hence there is a directed cycle C of free edges where every three consecutive vertices uvw along C span an edge uw belonging to F. Taking every second vertex of this cycle yields a cycle in F, i.e. a triangle. Hence C must be a 6-cycle, $C = u_0v_1u_2v_0u_1v_2u_0$ with two inscribed triangles from F, having the structure depicted in figure 2.3. All the vertices along this cycle are all in Y and two belongs to, say, Y_1. It follows from exchangeability that x_1 must be adjacent to at least four distinct vertices along the cycle C but this means x_1 is adjacent to two consecutive vertices, say u_2v_0. But then $T = x_1u_2v_0$ is a free triangle, since u_2v_0 is a free edge, which contradict (5). □

2.4 Proof of Theorem 2 (sketch)

The argument uses the bound (2) to find a free r-clique. We let F be $n-1$ vertex-disjoint r-cliques and define X and Y and the notion of exchangeable vertices and free sets in the analogous manner as above. Note that the degree condition in Theorem 2 implies that the complementary r-partite graph $K_r(n) \setminus G$, has maximum degree at most $\tilde{\Delta} = n/(1+l_{r-2}) - (r-1)l_{r-2}/(1+l_{r-2})$. It follows that the sets Y_i, $i = 0, 1, \ldots, r$, have at least $n' = n - \tilde{\Delta}$ elements and we consider an induced balanced subgraph H' on $r \cdot n'$ vertices. The minimum degree of H' is at least $(r-1)n' - \tilde{\Delta}$, which simplifies to $(r-1-1/l_{r-2})n' + r - 1$. Thus, if we let $H'' = H' \setminus F$ then H'' satisfies the bound in (2) so we find a K_r in H'' which then is necessarily free. □

2.5 Proof of Theorem 3 (sketch)

Let the parts of G be denoted $V_0, V_1, \ldots, V_{r-1}$, where indices are reduced modulo r. We let $F \subset G$ be $n-1$ vertex-disjoint admissible r-cycles and define $X = V \setminus V(F)$. We denote the element in $X \cap V_i$ by x_i . A vertex $u \in V_i$ is exchangeable if $d(x_i, F_u) = 2$, where F_u is the cycle in F containing u. Let Y be the set of exchangeable vertices. Then $Y_i = Y \cap V_i$ has at least $n/2+1$ elements. A set $S \subset Y$ is *free* if $|S \cap V_i| \leq 1$ and $F[S]$ does not contain any edges. It is easily checked that such a set can be exchanged with the corresponding subset of X.

Let H' be a balanced induced subgraph of $G[X \cup Y]$ with $n' = n/2 + 2$ vertices in each part. We have

$$d(v, H') \geq (3n/2) + 2 - 2(n - n') \geq n/2 + 4 = n' + 4.$$

We orient H' in the direction of increasing indices modulo r, and let $d^-(v)$ and $d^+(v)$ denote the in-degree and out-degree of v in $V(H')$, respectively. The degree condition implies that $d^+(v) \geq 4$ and hence we find a directed cycle $C = v_0v_1 \ldots v_{sr}$, $v_i \in Y_i$, in H' where each edge v_iv_{i+1} is a free edge. This cycle is schematically displayed in Figure 2.5.

We claim that we can find such a cycle C with $s = 1$, i.e. making just one round-trip. In this case it follows that $V(C)$ is a free set and we are

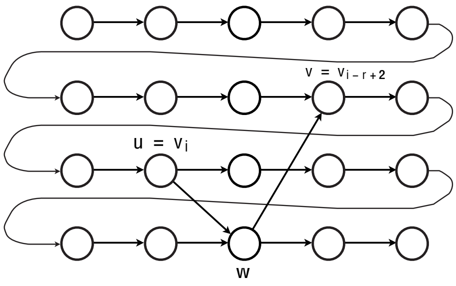


Figure 4: Finding a free cycle in H for $C_r(n)$.

done. Assume for contradiction that C is the smallest cycle and that $s \geq 2$. If along C there is some pair $u = v_i$, $v = v_{i-r+2}$, such that

$$d^+(u) + d^-(v) \geq n' + 3, \tag{15}$$

then there is a vertex $w \in V(H') \cap V_{i+1}$ which is adjacent to both u and v and such that both uw and wv are free edges in H. As is illustrated in Figure 2.5, we obtain the shorter cycle $C' = wv_{i-r+2}v_{i-r+3}\ldots v_i w$ of free edges. Inequality 15 must hold for some pair $u = v_i$, $v = v_{i-r+2}$ since

$$\sum_{i=0}^{rs-1} d^+(v_i) + d^-(v_{i-r+2}) = \sum_i d(v_i, H') \geq rs(n' + 4).$$

$\square$

2.6 Remarks

Another way of generalising to the case of cycles is to prescribe a local minimum degree condition: Let δ' be the minimal number of neighbours that a vertex $x \in V_i$ has in one of the sets V_{i-1} and V_{i+1}. (The "global minimum degree" is the smallest number of neighbours that a vertex $x \in V_i$ has in $V_{i-1} \cup V_{i+1}$.) It is proved in [Joh00] that $\delta' \geq \frac{2}{3}n + \sqrt{n}$ is sufficient to force a graph $G \subset C_3(n)$ to have a C_3-factor. It is also conjectured that the condition for a C_r-factor should be $\delta' \geq \frac{r+1}{2r}n + 1$ in this case. Hence this local minimum degree should depend on r contrary to the fact that the global minimum degree does not.

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