Extremal hypersgraphs and bounds for the Turan density of the 4-uniform K_5

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Abstract

In this paper we find, for $n \leq 16$, the maximum number of edges in a 4-uniform hypergraph which does not have the complete 4-uniform hypergraph on five vertices, $\mathcal{K}_5^4$, as a subgraph. Equivalently, we find all optimal $(n, n-4, n-5)$ covering designs for $n \leq 16$.

Using these results we find find a new upper bound for the Turan density of $\mathcal{K}_5^4$.

$$\pi(\mathcal{K}_5^4) \leq \frac{1753}{2380} = 0.73655\ldots$$

Finally we make some notes on the structure of the extremal 4-graphs for this problem and the conjectured extremal family.

1 Introduction

Given an r-uniform hypergraph H the Turan number $\mathrm{ex}(H, n)$ is the maximum number of edges in an r-uniform hypergraph on n vertices which is H-free, i.e. it does not have H as a subgraph. For 2-uniform (hyper)graphs the Turan numbers are well understood for the case of non-bipartite H, see e.g. [Bol04]. For $r \geq 3$ we currently have only scattered results and in particular $\mathrm{ex}(\mathcal{K}_t^r, n)$, the Turan numbers of the complete r-uniform hypergraph on t vertices, are only known for some small values of n, [Sid87].

It is easy to show that $\pi(H) = \lim_{n\to\infty} \frac{\mathrm{ex}(H,n)}{\binom{n}{r}}$, the Turan density of H, always exists and that $\frac{\mathrm{ex}(H,n)}{\binom{n}{r}}$ is a decreasing sequence [KNS64]. For $r \geq 3$ $\pi(\mathcal{K}_t^r)$ is not known for any $t \geq r+1$, and there are only a small number of cases where even a conjectured value exists.

For $\mathcal{K}_5^4$ Giraud [Gir90] found an ingenious construction for $\mathcal{K}_5^4$-free hypergraphs, based on Hadamard matrices, which implies that $\pi(\mathcal{K}_5^4) \geq \frac{11}{16}$ and Sidorenko later [Sid95] conjectured that this is in fact an equality. Sidorenko showed [Sid82] that $\pi(\mathcal{K}_5^4) \leq 0.749$ and for over 20 years this was the best bound known. Recently Lu and Zhao [LZ] found a new bound for $\pi(\mathcal{K}_{r+1}^r)$ which, among other things, improves Sidorenko's upper bound to $\pi(\mathcal{K}_5^4) \leq 0.744$.

In [dCKRM91] DeCaen et al. determined $\mathrm{ex}(\mathcal{K}_5^4, n)$ for $n \leq 10$ and also found the exact number of non-isomorphic Turan graphs for $n = 9, 10$. In this paper we report on a large scale computer search for extremal graphs for $\mathcal{K}_5^4$, i.e. $\mathcal{K}_5^4$-free hypergraphs with $\mathrm{ex}(\mathcal{K}_5^4, n)$ edges. We have constructed all non-isomorphic extremal graphs for $n \leq 16$ and found an upper bound on $\mathrm{ex}(\mathcal{K}_5^4, 17)$.

Using the new values of $\mathrm{ex}(\mathcal{K}_5^4, n)$ we find an improved upper bound of

$$\pi(\mathcal{K}_5^4) \leq \frac{1753}{2380} = 0.73655\ldots,$$

and make some further notes on the structure of the extremal graphs.

2 Giraud's construction

We will here give a brief description of Giraud's hypergraph family, mentioned in the introduction.

Let M be an $n \times n$ 0/1-matrix. We now define a hypergraph $G(M)$ with vertex set $V = V_r \cup V_c$, where V_r is the set of rows from M and V_c the set of columns. Given two rows a, b and two columns α, β we let $\{a, b, \alpha, \beta\}$ be an edge of $G(M)$ if the sum of the entries in the 2×2-submatrix defined by a, b, α, β is odd. We also let a four-tuple $\{a, b, c, d\}$ be an edge if either 1 or 3 of the elements are rows.

In order to maximise the number of edges in $G(M)$ we may use the matrix M obtained by replacing the -1:s in a Hadamard matrix by 0:s.

3 Constructing Extremal Graphs

The basic computational strategy we have used is to construct all non-isomorphic $\mathcal{K}_5^4$-free graphs on $n+1$ vertices and k edges by adding a new vertex, and edges incident to the new vertex, to the $\mathcal{K}_5^4$-free graphs on n vertices, and sufficiently many edges.

More precisely we use the bounds given by following simple lemma

Lemma 3.1. If G_1 is an $\mathcal{K}_5^4$-free 4-graph on n vertices and m edges then there exists an $\mathcal{K}_5^4$-free 4-graph G_2 on $n-1$ vertices and at least $m - \left\lfloor \frac{4m}{n} \right\rfloor$ edges, such that $G_2 = G_1 \setminus v$, for some $v \in V(G_1)$.

This lemma tells us both that the size of an extremal 4-graph on $n+1$ vertices can be bounded in terms of $\mathrm{ex}(n, \mathcal{K}_5^4)$. Furthermore, if we have found all $\mathcal{K}_5^4$-free 4-graphs on n vertices and e edges, for all $m - \lfloor 4m/n \rfloor \leq e \leq m$, then we can construct all $\mathcal{K}_5^4$-free 4-graphs on $n+1$ vertices and a given size m as follows:

1. Let S be the set of all $\mathcal{K}_5^4$-free 4-graphs on n vertices and e edges, for all $m - \lfloor 4m/n \rfloor \leq e \leq m$.

2. Given a 4-graph $G \in S$ let U_G be the set of all $\mathcal{K}_5^4$-free 4-graphs which can be constructed from G by adding a new vertex v to $V(G)$ and a set of $m - |E(G)|$ edges containing v.

3. Let $U = \cup_G U_G$ and let S' be the set of non-isomorphic 4-graphs in U.

4. S' is the set of all $\mathcal{K}_5^4$-free 4-graphs on n vertices and m edges.

That this simple procedure will produce all non-isomorphic $\mathcal{K}_5^4$-free 4-graphs on $n+1$ vertices and m edges follows directly from Lemma 3.1.

We found that if step 2 is done by a brute force combinatorial search this procedure is too slow for large n. Instead we formulated the extension step as an Integer programming problem which was then solved using the integer programming solver included in GNU's glpk-package [Mak]. Once an extension of a 4-graph G had been found we added a new linear inequality which excludes only that particular solution and the process was repeated until the new integer program had no solutions.

Finally the isomorphism reduction in step 3 was done using Brendan McKay's graph automorphism program Nauty [McK81].

4 Results

Using the method described in the previous section we were able to find the set of extremal 4-graphs for $n \leq 16$. For $n = 17$ we could not find $\mathrm{ex}(n, \mathcal{K}_5^4)$ but since none of the 4-graphs on 16 vertices could be extended to a 4-graph on 1754 edges, or more, we could reduce the upper bound on $\mathrm{ex}(17, \mathcal{K}_5^4)$. The number of extremal 4-graphs and their sizes are shown in Table 4. These, and other Turan hypergraphs, are available on the web [Mar], as are the equivalent covering designs.

n	size	bound	opt-0	opt-1	opt-2	opt-3	opt-4
6	12	12	1	3			
7	28	28	1	1			
8	56	56	1	1	5	48	
9	96	100	3	51	2205		
10	160	160	1	3	94	3240	123275
11	246	251	3	128	10322		
12	369	369	3	35	2960	305900	
13	530	533	1	22	3223		
14	742	742	1	7	945	204202	
15	1008	1011	1	5			
16	1344	1344	1	2	97		
17	$\leq$1753	1757					

Figure 1: Near extremal 4-graphs for $\mathcal{K}_5^4$. The columns are as follows: size: the number of edges in the extremal 4-graphs, bound: The upper bound on the number of edges given by the extremal 4-graph with one vertex less, opt-a: The number of non-isomorphic $\mathcal{K}_5^4$-free 4-graphs with a edges less than the extremal 4-graphs.

4.1 Density and Stability

Using the upper bound on $\mathrm{ex}(17, \mathcal{K}_5^4)$ and the fact that $\frac{\mathrm{ex}(H,n)}{\binom{n}{r}}$ is a decreasing sequence we find the following upper bound on the Turan density of $\mathcal{K}_5^4$.

Corollary 4.1. $\pi(\mathcal{K}_5^4) \leq \frac{1753}{\binom{17}{4}} = 0.7365546\ldots$

For all $n \leq 16$ $\mathrm{ex}(n, \mathcal{K}_5^4)$ is exactly that given by Giraud's construction [Gir90]. That this was true for $n \leq 10$ was observed in [Sid95]. As the table shows this construction does in fact give the unique extremal 4-graph for $13 \leq n \leq 16$. We would like to venture a strengthening of Sidorenko's conjecture about the asymptotic optimality of Giraud's construction.

Conjecture 4.2. For $n \geq 12$ the only $\mathcal{K}_5^4$-free hypergraphs on $\mathrm{ex}(n, \mathcal{K}_5^4)$ edges are those given by the construction from [Gir90].

An inspection of the $\mathcal{K}_5^4$-free 4-graphs on $\mathrm{ex}(n, \mathcal{K}_5^4) - 1$ edges shows that except for $n = 9$ and $n = 12$ all such hypergraphs are subgraphs of at least one extremal 4-graph on the same number of vertices. For $n = 9$ there are 18, and for $n = 12$ there are 2, additional 4-graphs of this size.

Examining the $\mathcal{K}_5^4$-free hypergraphs on $\mathrm{ex}(n, \mathcal{K}_5^4) - 2$ edges we find that for $n = 13$ all but 6, and that for $n = 14$ and 16 all, are subgraphs of the unique extremal 4-graph. In view of this we expect a stability version of Conjecture 4.2, analogous to Simonovits's theorem [Sim68] to hold as well, showing that the near-extremal hypergraphs are also close to Giraud's construction.

4.2 The numbers of Turan graphs

Given that the extremal 4-graph is unique for $n = 13 \dots 16$ it might be tempting to expect this to hold for all larger n as well. However if Giraud's construction is optimal this will not be the case.

Let us recall that two Hadamard matrices M_1 and M_2 are equivalent, denoted $M_1 \simeq M_2$, if M_2 can be obtained from M_1 by a sequence of the following operations

1. Permuting rows and/or columns.
2. Transposition.
3. changing the sign of all entries in a row/column.

Via operation 3 we may always bring a Hadamard matrix to its standard form, i.e. with only positive entries in the first row and column. We will now show

Theorem 4.3. Given two Hadamard matrices M_1 and M_2 in standard form, $M_1 \simeq M_2$ if and only if $G(M_1)$ is isomorphic to $G(M_2)$

Proof. Recall that the vertex set of $G(M_1)$ is partitioned into two independent sets V_r and V_c. A row permutation of M_1 is equivalent to a permutation of V_r and a column permutation to one of V_c. Transposing M_1 is equivalent to interchanging V_r and V_c. Applying a sign change operation on M_1 does not affect $G(M_1)$.

It thus immediate that if $M_1 \simeq M_2$ then $G(M_1)$ is isomorphic to $G(M_2)$.

However, for Hadamard matrices in standard form the mapping from M to $G(M)$ is a bijection. This can be seen by noting that the second row of the matrix can be found by considering the possible edges using the first two rows of M. The presence or not of the edge using the first two rows and columns determines the sign of the only unknown matrix position in the underlying 2×2-submatrix, and once that is found the other positions in that row can be found in the same way, and the the remaining rows.

Thus an isomorphism between $G(M_1)$ and $G(M_2)$ will also define an equivalence between G_1 and G_2. $\square$

For small orders there exist only one Hadamard matrix of each order, see e.g. chapter 7 of [HSS99], and thus Giraud's construction give rise to a unique 4-graph. However, at order 16 there are 5 non-equivalent Hadamard matrices, and it is possible to construct sequences of Hadamard matrices, of certain orders, with more than one matrix per order. So if Sidorenko's conjecture is true the extremal 4-graph will not be unique for infinitely many $n \geq 32$.

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