# Complete Minors in Cubic Graphs with few short Cycles and Random Cubic Graphs

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ABSTRACT. We first prove that for any fixed $k$ a cubic graph with few short cycles contains a $K_k$-minor. This is a direct generalisation of a result on girth by Thomassen. We then use this theorem to show that for any fixed $k$ a random cubic graph contains a $K_k$-minor asymptotically almost surely.

## 1. Introduction

During the last 30 years minors has been one of the most active topics in graph theory. Much of this interest has stemmed from the result by Kruskal [7] saying that the trees are well quasi ordered under the ordering induced by the relation "$H$ is a minor of $G$" and its culmination(?) with the corresponding result, the "Minor Theorem", for general graphs by Robertson and Seymour [10].

Of course this was not the first use of minor in graph theory. The best known early use of minors is of course Kuratowski's characterisation of planar graphs [8]. After this comes the now classic conjecture by Hadwiger [4],

**Conjecture 1.1** (Hadwiger's Conjecture)**.** *If a graph $G$ has chromatic number $k$ then $G$ has a $K_k$-minor.*

As part of both the minor theorem and the attempts to understand Hadwiger's conjecture one has been interested in other conditions which implies that a graph has some graph $H$ as a minor, and especially the case when $H$ is a complete graph. One could say that the first such conditions are the results of Turán [14], and Erdős and Stone [2], showing that a high enough average degree implies the existence of both a complete graph and a general graph $H$ as a subgraph. However for minors rather that subgraphs Mader [9] showed that $8k \log k$ edges suffice to get a $K_k$-minor, this was later refined to $ck\sqrt{\log k}$ by Kostochka [6], and independently by Thomason [11] who later also determined the constant $c$ [12]. At the same time as Mader proved his result Wagner [15] found a condition closer to that in Hadwiger's conjecture, namely that a chromatic number of at least $2^k$ gives a $K_k$-minor. Finally, and closer to what will be done in the current paper, Thomassen [13] studied the effect of girth on minors and found that for graphs with minimum degree 3 a high enough girth will also force a $K_k$-minor.

Turning to random graphs instead Bollobas, Catlin and Erdős found that in random graphs from $G(n,p)$, for a suitable $p$, the largest $k$ for which one expects to find a $K_k$-minor is bounded by $\frac{n}{(\sqrt{\log n})+4} \leq k \leq \frac{n}{(\sqrt{\log n})-1}$. Together with the results on colouring by Grimmet and McDiarmid [3] this shows that for a random graph one expects Hadwiger's conjecture to hold.

In the current paper we focus on random cubic graphs. We prove a generalisation of Thomassen's theorem on girth and then use it together with the results of Bollobas and Wormald to prove that for any given $k$ a large enough cubic graph $G$ contains a $K_k$-minor with high probability. We also give conjecture regarding how large $k$ can be as a function of the order of $G$.

## 2. Results

First some necessary definitions.

**Definition 2.1.** We let $\mathcal{G}_{n,d}$ denote the probability space of all $d$-regular simple graphs on $n$ vertices equipped with the uniform probability measure.

**Definition 2.2.** We say that a graph in $\mathcal{G}_{n,d}$ has a given property asymptotically almost surely (a.a.s) if the probability that a graph from $\mathcal{G}_{n,d}$ has the property tends to 1 as $n \to \infty$.

We can now state our first theorem.

**Theorem 2.3.** *Given integers $n_c$ and $k$ there exist an $N_0 = N_0(n_c, k)$ such that every cubic graph $G$ with at most $n_c$ cycles of length shorter than $g = 2k-3$ and $|G| \geq N_0$ has a minor of average degree greater than $\frac{k}{6} - 1$.*

This iss a direct generalisation of a theorem of Thomassen, concerning graphs with given girth [13]. The proof below follows Thomassen's proof closely, adding the part necessary to give a lower bound of the order of the minor.

*Proof.* We first assume that $G$ is connected. Let $n = |G|$. Now consider a partition $A = (A_1, A_2, \ldots, A_m)$ of the vertices of $G$ into connected subgraphs with $|A_i| \geq k-2$. Such a partition exists for $m = 1$ and so we can consider a partition maximising $m$.

First of all we can draw the conclusion that $|A_i| < 3(k-2)$. If not we could just split $A_i$ into two smaller connected components, thereby violating the maximality of $m$. That $A_i$ can be split in this way follows from the lemma below by considering a spanning tree.

Next let $A^g_1, A^g_2, \ldots, A^g_p$ be the $A_i$'s which do not contain vertices belonging to cycles of length less than $2k-3$, and let $A^b_1, A^b_2, \ldots, A^b_l$ be the remaining $A_i$'s. Clearly $l < gn_c$.

We will next show that $G(A^g_i)$ is a tree. Let $T_i$ be a spanning tree of $G(A_i)$ and assume that $T_i \neq G(A_i)$. Then there must be an edge in $E(G(A_i)) \setminus E(T_i)$ such that $T_i \cup e$ contains a cycle of length at least $2k-3$. Thus we can find another edge $e'$ in $T_i$ such that $T_i \setminus e'$ has two components of cardinality at least $k-2$, thus violating the maximality of $m$ once more.

Third we show that no two trees $T_i$ $T_j$, corresponding to some $A^g_i$ and $A^g_j$, are connected by more than two edges. Assume that there are three edges $e_1, e_2, e_3$ connecting $T_i$ and $T_j$. Then we can find two vertices $u \in T_i$ and $v \in T_j$ such that there are three internally vertex disjoint paths $P_1, P_2, P_3$ in $T_i \cup T_j \cup e_1 \cup e_2, \cup e_3$, each with endpoints $u$ and $v$. Each pair of paths form a cycle of length at least $2k-3$, if not we would be in an $A^b_j$, and so at least two of the paths, say $P_1$ and $P_2$, must have length at least $k-1$. Now let $P'_i \subset P_i$, $i = 1, 2$, be two subpaths of length $k-3$ using only internal vertices of the $P_i$. Put $P'_3 = (P_1 \cup P_2 \cup P_3) \setminus (P'_1 \cup P'_2)$ and $C = P_1 \cup P_3$. Now $P'_3$ is connected and we find that

$$|P'_3| \geq |C \setminus P'_1| \geq g - (k-2) = k-1.$$

Thus both $P'_1, P'_2$ and $P'_3$ are connected sets with more than $k-2$ vertices. Now by redistributing the remaining vertices, if any, of $T_i \cup T_j$ we have found a partition violating the maximality of $m$ again.

Now we form a new graph $G^*$ by contracting each $A_i$ to a vertex and removing multiple edges in order to get a simple graph. Since each $A^g_i$ is a tree the vertex in $G^*$ corresponding to $A^g_i$ will have degree at least $\frac{k}{2}$ after reducing double edges.

Finally $G^*$ must have at least $\frac{1}{2}\left(\frac{n}{3k-6} - l\right)\frac{k}{2}$ edges and the average degree is at least

$$\left(\left(\frac{n}{3k-6} - l\right)\frac{k}{2}\right)\frac{1}{l+p} \geq \left(\left(\frac{n}{3k-6} - l\right)\frac{k}{2}\right)\frac{1}{\frac{n}{k-2}} \geq \frac{k}{6} - \frac{k^2 n_c g}{2n}.$$

Since $k$ and $n_c$ are fixed this fraction will be greater than $\frac{k}{6} - 1$ for $n$ greater than some $N_0$.

If $G$ is not connected we find that unless some component has girth at least $g$ the number of components is bounded by $n_c$. Thus we can apply the previous reasoning to the component with smallest $n_c$ and we are done. $\square$

As can be seen in the proof $N_0$ is linear in $n_c$ if $G$ is connected, quadratic in $n_c$ if $G$ is unconnected, and cubic in $k$.

**Lemma 2.4.** *Let $T$ be a tree on $3t$ vertices, with maximum degree at most 3. Then the vertex set of $T$ can be partitioned into two trees $T_1$ and $T_2$ such that $|T_i| \geq t$.*

*Proof.* We will make a proof by contradiction. Assume that $T$ is a tree for which the statement fails.

Given an edge $e \in E(T)$ we have that $T \setminus e$ consists of two trees $T_{1,e}$ and $T_{2,e}$ and by assumption we have that one of then, say $T_{1,e}$, has less than $t$ vertices. Let $e = (u, v)$ be chosen such that the order of $T_{1,e}$ is maximal and let $u$ be the vertex in $e$ which belongs to $T_{2,e}$.

Now $u$ must have degree 3, if not there would be an edge $e' = (u, w)$ such that $|T_{1,e'}| = |T_{1,e}|+1$, and $T_{2,e'}$ must still have at least $3t-(t-1) = 2t+1$ vertices, contradicting our choice of $e$.

Now the vertex $u$ has degree 2 in $T_{2,e}$ an so $T_{2,e} \setminus u$ will consist of two subtrees of $T_{2,e}$, call them $T_{1,u}$ and $T_{2,u}$. Since the order of $T$ is $3t$ and the order of $T_{1,e}$ is less than $t$ we must have that at least one of $T_{1,u}$ and $T_{2,u}$, say $T_{1,u}$, has order at least $t$. But then $T_{1,e} \cup u \cup T_{2,u}$ will be larger than $T_{1,e}$ contradicting our choice of $e$.

□

We now come to our probabilistic theorem.

**Theorem 2.5.** *Let $k$ be a fixed integer. A graph in $\mathcal{G}_{n,3}$ has a $K_k$-minor a.a.s.*

**Corollary 2.6.** *Given a graph $H$, a cubic graph has an $H$-minor a.a.s.*

In order to prove this we need a result by Kostochka, and independently Thomason, connecting complete minors and average degree.

**Theorem 2.7** ([6] [11])**.** *There exists a $c$ such that for every $k$, every graph of average degree $d \geq ck\sqrt{\log k}$ has a $K_k$-minor.*

The value of $c$ has recently been determined closely by Thomason [12].

And as our final ingredient we need a result by Bollobas and Wormald on the number of short cycles in random cubic graphs.

**Theorem 2.8** ([16] [1])**.** *Let $X_i$ be the number of cycles of length $i$ in a graph in $\mathcal{G}_{n,3}$. For a fixed $k$ $X_3, X_4, \ldots, X_k$ are asymptotically independent Poisson random variables with means $\lambda_i = \frac{2^i}{2i}$.*

We now have all we need in order to prove the theorem.

*Proof of theorem 2.5.* Let $k$ be given and choose $r > ck\sqrt{\log k}$, where $c$ is the constant in 2.7. By theorem 2.8 the expected number of cycles of length less than $6(2r-3)$ is asymptotically a Poisson random variable with expectation and variance $\lambda \leq 2^{6(2r-3)}$, just compute the sum of the individual random variables.

By Chebyshew's inequality we find that asymptotically a cubic graph has more than $\lambda + b\lambda^{\frac{1}{2}}$ cycles of length less than $6(2r-3)$ with probability at most $\frac{4}{b^2}$ and so by the lemma lacks a minor of average degree $r$ with probability at most $\frac{4}{b^2}$. Since the $X_i$'s are only asymptotically Poisson distributed we include the factor 4 to allow for asymptotically vanishing deviations from this distribution.

Using theorem 2.7 we find that asymptotically a cubic graph lacks a $K_k$-minor with probability less than $\frac{4}{b^2}$ as well. Since this holds for all $b$ it also holds with probability one.

$\square$

## 3. Some thoughts

Fist we can note that using contiguity type methods the theorem could be extended from the cubic case to $k$-regular graphs in general.

The methods used here are quite crude and the probability for an $H$-minor is much higher than shown here. A natural follow up to the present result would of course be,

**Problem 3.1.** *Given a uniformly distributed random $d$-regular graph on $n$ vertices, what is the largest $k$ for which there is $K_k$-minor in $G$ with probability at least $\frac{1}{2}$.*

I would further like to conjecture that the case of a random cubic graph is not essentially worse than that of a cubic graph of high girth. Although of course a graph can not have girth of order $\sqrt{n}$.

**Conjecture 3.2.** *There exists a constant $c > 0$ such that if $k = c\sqrt{n}$ then the probability that a uniformly distributed cubic graph has a $K_k$-minor tends to 1.*

We may note that using standrard probabilistic methods it is not hard to get quite close to the conjectured value of $c$, but pushing it to a pure $\mathcal{O}\left(\sqrt{n}\right)$ does not seem to be easy.

Next let us consider corollary 2.6 for the case when $H$ too is a cubic graph. In this case an $H$-minor in a cubic graph $G$ will correspond to a subgraph of $G$ isomorphic to a subdivision of $H$. Looking a the pairing model for random regular graphs the probability for finding $H$ itself as a subgraph of $G$ is $\mathcal{O}\left(n^{-k}\right)$, with $n = |G|,\ 2k = |H|$, see [5]. As we can see this probability tends to zero quite quickly. However there are $(3k)^m$ different ways to subdivide the edges of $H$, introducing $m$ new vertices, and we find that the expected number of such subgraphs of $G$ is $\mathcal{O}(n^{-k}(3k)^m)$. Thus for $m < k\frac{\log n}{\log 3k}$ this expectation tends to zero and so an $H$-minor of $G$ will a.a.s contain at least $k\frac{\log n}{\log 3k}$ vertices. This makes it all the more remarkable that by the results of Robertson and Seymour we can find an $H$-minor of $G$, should one exist, using an algorithm with running time $\mathcal{O}(n^3)$.

A more philosophical note is that one should take some care in the use of excluded minor results. As shown here one can never expect any large proportion of all cubic graph to exclude some given minor. When the property one is examining is completely characterised by the exclusion of some set minors, as it is for planarity and the cycle cover property, this is of course unavoidable, and so the property is in some sense rare. But for cases where one expects a property to be common, or even hold for all graphs, results of excluded minor type are an unlikely, although perhaps not impossible, road to the desired goal.

## Acknowledgement

I would like to thank Robert Johansson for reading several early versions of the manuscript and giving numerous useful comments.

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