# Exact and Approximate Compression of Transfer Matrices for Graph Homomorphisms

Per Håkan Lundow and Klas Markström

February 23, 2006

**Abstract**

The aim of this paper is to extend the previous work on transfer matrix compression in the case of graph homomorphism. For $H$-homomorphisms of lattice-like graphs we demonstrate how the automorphisms of $H$, as well as those of the underlying lattice, can be used to reduce the size of the relevant transfer matrices.

As applications of this method we give currently best known bounds for the number of 4 and 5-colourings of the square grid, and the number of 3 and 4-colourings of the 3-dimensional cubic lattice.

Finally we also discuss approximate compression of transfer matrices.

## 1 Introduction

Transfer matrices is a standard tool in various branches of mathematics. In enumerative combinatorics they have been used for a long time to solve counting problems which can be described using graph homomorphisms. In statistical physics transfer matrices have been a well used tool for the computation of partition functions of various spin models. In ergodic theory transfer matrices are used to describe the behaviour of a class of dynamical system known as subshifts of finite type. Recently the many similarities between these uses have been put on a firm mathematical ground. That the calculation of entropies for $\mathbb{Z}^d$-subshifts of finite type are equivalent to counting graph homomorphisms from $\mathbb{Z}^d$ into some graph $H$ was demonstrated in [5]. In [9] it was proven that counting weighted graph homomorphisms is equivalent to computation of the partition function for statistical physics models satisfying a condition known as reflection positivity. The study of phase transitions in statistical physics models has also begun to be studied in the language of homomorphisms [3].

A general limitation for all these applications is that transfer matrices tend to grow fast with the size of the system considered, thus limiting the size of the system one can work with. A recent development in this area is the use of automorphisms of the underlying graphs to reduce the size of the matrices. This was done for a special case in [4] and was developed as a general method in [10]. Our aim here is first to show how this method, now called transfer matrix compression, in many cases can be taken even further than earlier applications and also discuss how one can make even greater size reductions if one is ready to settle for bounds of the computed entropies rather than exact values. For many applications the latter is sufficient.

Let us put things on a firmer ground. A homomorphism $\phi$ from a graph $G$ to a graph $H$, which may have loops, is a mapping which preserves adjacencies, i.e. if $(x,y) \in E(G)$ then $(\phi(x),\phi(y)) \in E(H)$. The set of all homomorphisms from $G$ to $H$ is denoted $\mathsf{Hom}(G,H)$. We say that $H$ is a weighted graph if there are two functions $\alpha_H : V(H) \to F$ and $\beta_H : E(H) \to F$, where $F$ is a ring. Given a weighted graph $H$ we assign a weight $w(\phi)$ to each homomorphism $\phi$ from $G$ to $H$

$$w(\phi) = \prod_{x \in V(G)} \alpha_H(\phi(x)) \prod_{xy \in E(G)} \beta_H(\phi(x),\phi(y)) \tag{1}$$

Let us give a few examples:

**Example 1.1.**

1. $H = K_q$, $\alpha = \beta = 1$, corresponds to ordinary proper $q$-colourings.
2. Let $H$ be a $K_2$ with a loop on one vertex, let the edge have weight 1, the loop weight $t$ and $\alpha = 1$. For $t = 1$ the homomorphisms correspond to independent sets and for general $t$ we have the so called hard-core lattice gas model.
3. If $H$ is a $K_q$ with loops on every vertex and the weights are $t^{-1}$ on the loops, $t$ on the ordinary edges and 1 on the vertices we have the $q$-state Potts model.
4. If we take the previous example with $q = 2$ and put a weight $s$ on one vertex of $H$ and $s^{-1}$ on the other vertex, we have the Ising model with an external field.

If we have a model where we also want to put colours or "spins" on the edges of $G$, e.g. when considering matchings, or on both edges and vertices, we can instead consider the line-graph or total-graph of the underlying graph $G$ respectively. It is also straightforward to generalise these concepts to hypergraphs if one wants to consider interaction between larger sets of vertices.

We next define a weighted counter for these homomorphisms

$$Z(G,H) = \sum_{\phi \in \mathsf{Hom}(G,H)} w(\phi)$$

If all weights on $H$ are just 1 this will be exactly $|\mathsf{Hom}(G,H)|$. In spin models the weights are often taken to be of the form $e^K$ for a parameter $K$ which is interpreted as a temperature, and then $Z(G,H)$ is called the partition function of the model. In ergodic theory $Z(G,H)$ is called the pressure of the subshift described by $G$ and $H$. In most applications the aim is either to compute $Z(G,H)$ when $F$ is a ring of polynomials, as in [7], or to determine how fast $Z(G_n,H)$ grows when $F=\mathbb{R}$ and $G_n$ is some sequence of graphs, see e.g. [5, 4].

# 2 Polygraphs and Transfer Matrices

Transfer matrices are most useful for computation within a class of graphs known as polygraphs. This class was introduced in [1], where the transfer matrices were used to compute matching polynomials. A polygraph $\mathcal{G}$ is defined by a set of disjoint graphs $G_1, G_2, \ldots, G_m$ and a set of binary relations $\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_m$, where $\Upsilon_i \subset V(G_i) \times V(G_{i+1})$. The vertex set of $\mathcal{G}$ is $\cup_i V(G_i)$ and the edge set is

$$E(\mathcal{G}) = \cup_i E(G_i) \cup_i \Upsilon_i$$

If all $G_i = G$ and $\Upsilon_i = \Upsilon$ for all $i = 1, \ldots, m$ we write the corresponding polygraph as $\mathcal{G}(G, \Upsilon, m)$.

Given a polygraph $\mathcal{G}$ and a weighted graph $H$ we can compute $Z(\mathcal{G}, H)$ using a sequence of transfer matrices. We define a matrix $M(i)$ for going from $G_i$ to $G_{i+1}$ as follows.

Let $\Phi(G_i)$ denote the set of restrictions of all homomorphisms in $\mathsf{Hom}(\mathcal{G}, H)$ to $G_i$. We call a member of $\Phi(G_i)$ a *state* on $G_i$. Now let the rows of $M(i)$ be indexed by the states on $G_i$ and the columns by the states on $G_{i+1}$. We set $M_{x,y}(i) = 0$ if there is no homomorphism $\phi \in \mathsf{Hom}(\mathcal{G}, H)$ such that $\phi|_{G_i} = x$ and $\phi|_{G_{i+1}} = y$. If there exists such a $\phi$ we set $M_{x,y}$ equal to the contribution to the weight $w(\phi)$ of the edges in $\Upsilon_i$ and the edges and vertices of $G_{i+1}$. We also define an associated vector $\eta$. The position in $\eta$ corresponding to the row $x$ is set equal to the weight of the partial homomorphism $x$.

The partition function is now given by

$$Z(\mathcal{G}, H) = \eta \left( \prod_i M(i) \right) \mathbf{1} \tag{2}$$

One can also consider cyclic polygraphs where the last relation $\Upsilon$ connects $G_m$ to $G_1$. In this case the partition function is given by the trace of the transfer matrix product,

$$Z(\mathcal{G}, H) = \mathrm{Tr} \left( \prod_i M(i) \right) \tag{3}$$

# 3 Exact Compression of Transfer Matrices

Henceforth we will assume that our polygraphs are on the form $\mathcal{G}(G, \Upsilon, m)$. Most of what follows can be adapted to general polygraphs as well. Let us recall that given an $N \times N$ matrix $M$ a partition $\mathcal{X} = \{X_1, X_2, \ldots, X_r\}$ of $\{1, ..., N\}$ is called an equitable partition if $\sum_{l \in A_j} M(i_1, j) = \sum_{l \in X_j} M(i_2, j)$ when $i_1, i_2 \in X_i$.

Given a partition $X$ of the states on $G$ we define the compressed transfer matrix for $Z(\mathcal{G}, H)$ to be

$$C_{\mathcal{X}}(i,j) = \sum_{l \in X_j} M(k,l), \quad k \in X_i, \quad i,j = 1 \ldots r \tag{4}$$

The main theorem of [10] can be stated as

**Theorem 3.1.** *If $X$ is an equitable partition of $M$ then*

$$C^n_{\mathcal{X}}(i,j) = \sum_{l \in X_j} M^n(k,l), \quad k \in X_i, \quad i,j = 1 \ldots r \tag{5}$$

This has the following corollary

**Corollary 3.2.** *Let $\eta$ be the vector of length $N$ whose i:th entry is $|X_i|\ w(x_i)$, where $x_i \in X_i$. Then*

$$Z(\mathcal{G}, H) = \eta \left( \prod_i M(i) \right) \mathbf{1}$$

The main consequence of these results, which has been used in [10], [4], [5], is

**Corollary 3.3.** *If $M$ is a transfer matrix and the partition $\mathcal{X}$ consists of orbits on the set of states on $G$ under the automorphism group $Aut(G)$, then $\mathcal{X}$ is equitable.*

This corollary lets us make use of automorphisms of $G$ to compress our transfer matrices and for graphs with reasonably large automorphism group, such as cycles, the reduction in size can be substantial.

**Example 3.4.** Let us look at the transfer matrix for $\mathsf{Hom}(G(P_3, Id, n), K_3)$, i.e. 3-colourings of the graph $P_3 \times P_n$.

There are 12 states on $P_3$ and the only nontrivial member of $\mathrm{Aut}(P_3)$ is a reflection in the midpoint. If we use 1,2,3 to denote colours we find that there are 9 orbits: $\{\{121\}, \{212\}, \{313\}, \{131\}, \{232\}, \{323\}, \{123, 321\}, \{132, 231\}, \{213, 312\}\}$

Here we get the following $9 \times 9$ matrix, instead of a $12 \times 12$ matrix,

$$C = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \qquad \eta = \{1,1,1,1,1,1,2,2,2\}$$

Even though we have reduced the side of the matrix by one quarter we still have a fairly sparse matrix.

From algebraic graph theory [6] we know that if $G$ is any graph and $X$ is a partition of its vertices into orbits under $\mathrm{Aut}(G)$ then the corresponding partition is equitable. If we choose to interpret the transfer matrix $M$ as the adjacency matrix of a weighted graph, then what we did in Corollary 3.3 can be interpreted as using the subgroup of $\mathrm{Aut}(M)$ which is induced by $\mathrm{Aut}(G)$ to partition $M$. However, $\mathrm{Aut}(M)$ has an even larger subgroup induced by $\mathrm{Aut}(G) \times \mathrm{Aut}(H)$. Here $\mathrm{Aut}(H)$ is assumed to preserve the weights on $H$ as well as the adjacencies.

**Corollary 3.5.** *Let $X$ be a partition of $M$ given by the orbits of $Aut(G) \times Aut(H)$ then $X$ is equitable.*

When the graph $H$ is highly symmetric, such as in the case of proper colourings of $G$ or the partition function of the Potts model, the extra reduction in size achieved here can be remarkable.

**Example 3.6.** If we use the automorphism group of $K_3$ as well in Example 3.4 we now only find two orbits, $\{\{121, 212, 313, 131, 232, 323\}, \{123, 321, 132, 231, 213, 312\}\}$, and the compressed transfer matrix has side 2:

$$C = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \quad v = \{6, 6\}$$

# 4 Application to the asymptotic number of $q$-colourings of lattices

The number of proper $q$-colourings of an $n \times n$ square grid, or $P_n \times P_n$, is known to grow exponentially as a function of $n^2$. We denote the basis for this exponential growth by $\lambda^s(q)$. This is a quantity which is of interest both in enumerative combinatorics and statistical physics. In the latter case $\log \lambda^s(q)$ is seen as the ground state entropy of the much studied antiferromagnetic Potts model on the square grid, see e.g. [11] for a survey.

For 3-colourings Lieb [8] found the exact asymptotic value of $\lambda^s(3) = \left(\frac{4}{3}\right)^{\frac{3}{2}}$ but for larger $q$ the value of $\lambda^s(q)$ is still unknown. As a full-scale example of our methods we now look at the compression of transfer matrices for $q$-colourings of the square grid for $q = 4, 5$, for the cubic grid $P_n \times P_n \times P_n$ for $q = 3, 4$, and their use in getting bounds for $\lambda^s(q)$.

It is known that the maximum eigenvalues $\theta_1(k)$ and $\theta_2(k)$ of the transfer matrices for $\mathsf{Hom}(G(P_k, Id, n), K_q)$ and $\mathsf{Hom}(G(C_k, Id, n), K_q)$ respectively, can be used to give upper and lower bounds for $\lambda^s(q)$, see e.g. [5] for a general treatment. In particular

$$\frac{\theta_1(k+1)}{\theta_1(k)} \leq \lambda^s(q) \leq \theta_2(2k)^{\frac{1}{2k}} \tag{6}$$

So we can bound $\lambda^s(q)$ by computing $\theta_1(k)$ for consecutive $k$ and $\theta_2(k)$ for even $k$.

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|
| $N_1$ | 36 | 108 | 324 | 972 | 2916 | 8748 | 26244 | 78732 | 236196 |
| $N_2$ | 24 | 54 | 180 | 486 | 1512 | 4374 | 13284 | 39366 | 118584 |
| $N_3$ | 2 | 4 | 10 | 25 | 70 | 196 | 574 | 1681 | 5002 |
| $N_1$ | 24 | 84 | 240 | 732 | 2184 | 6564 | 19680 | 59052 | 177144 |
| $N_2$ | 4 | 21 | 24 | 92 | 156 | 498 | 1096 | 3210 | 8052 |
| $N_3$ | 1 | 3 | 2 | 9 | 10 | 34 | 57 | 169 | 366 |

Table 1: Transfer matrix sizes for 4-colourings. the upper three rows are for transferring a path, and the lower three for a cycle.

In Table 1 we have given the size of the transfer matrix for $\mathsf{Hom}(G(P_k, Id, n), K_3)$ and $\mathsf{Hom}(G(C_k, Id, n), K_3)$. Here $N_1$ denotes the side of the uncompressed transfer matrix, $N_2$ the side when the automorphism group of $P_k$ and $C_k$ respectively were used, and $N_3$ the size when the automorphism group of $K_3$ was used as well. The effect of the larger automorphism group of the cycle is well visible, as is the gain from including the automorphism group of $K_4$ in the compression step.

For small $k$ the transfer matrices and eigenvalues were computed first with a `Mathematica` program and also with a Fortran 90 program. For larger $k$ the Fortran 90 program was run on a linux cluster. In Table 2 of Appendix A we have collected the computed eigenvalues. The higher precision values for small $k$ are due to the `Mathematica` program.

Using these eigenvalues and inequalities 6 we find the following bounds for $\lambda^s(4)$:

$$2.336056640723116 \leq \lambda^s(4) \leq 2.33606820555777 \tag{7}$$

To our knowledge these are currently the best rigorous bounds for $\lambda^s(4)$. In [2] the first terms of a series expansion in $\frac{1}{q-1}$ for $\lambda^s(q)$ was obtained and using this series it was estimated that

$$\lambda^s(4) = 2.336056641 \pm 0.000\ 000\ 001,$$

with a heuristic error bound, an estimate which fits in just above our lower bound.

In the same way we computed the corresponding eigenvalues for 5-colourings, given in Table 3 of Appendix A. The bounds so obtained for $\lambda^s(5)$ are

$$3.2504049231640764 \leq \lambda^s(5) \leq 3.250407145038276 \tag{8}$$

For the growth rate $\lambda^c(q)$ of the number $q$-colourings of the cubic lattice $P_k \times P_k \times P_k$ there are no exact results known. In [2] series estimates were also given for $\lambda^c(3)$ and $\lambda^c(4)$, however these estimates were based on much shorter series than those for the square grid and were given as

$$\lambda^c(3) = 1.4435 \pm 0.0005$$

$$\lambda^c(4) = 2.043 \pm 0.001$$

In order to compute bounds for the cubic lattice we can make use of the observations that $\lambda^c(q)$ is greater than the maximum eigenvalue for $\mathsf{Hom}(G(P_k \times C_\ell, Id, n), K_q)$, since colourings of these graphs can be extended to periodic colourings of $\mathsf{Hom}(G(P_k \times P_{\ell\ t}, Id, n), K_q)$ for all $t$. Likewise $\lambda^c(q)$ is less than the maximum eigenvalue for $\mathsf{Hom}(G(P_k \times P_\ell, Id, n), K_q)$, since the number of colourings is submultiplicative.

As before we can get lower bounds for the maximum eigenvalue of $\mathsf{Hom}(G(P_k \times C_\ell, Id, n), K_q)$ for a fixed $\ell$ by computing the maximum eigenvalues for consecutive $k$, and for each value of $\ell$ we will get lower bound for $\lambda^c(q)$. Similarly we can get upper bounds for the maximum eigenvalue of $\mathsf{Hom}(G(P_k \times P_\ell, Id, n), K_q)$ by computing the maximum eigenvalue of $\mathsf{Hom}(G(C_k \times P_\ell, Id, n), K_3)$ for even $k$. These eigenvalues are in turn bounded from above by the maximum eigenvalue of $\mathsf{Hom}(G(C_k \times C_\ell, Id, n), K_3)$, for even $k$ and $\ell$.

For $\lambda^c(3)$ our best bounds comes from the eigenvalues of $\mathsf{Hom}(G(C_6 \times C_6, Id, n), K_q)$ and $\mathsf{Hom}(G(P_k \times C_4, Id, n), K_q)$, for $\lambda^c(4)$ the bounds were achieved by $\mathsf{Hom}(G(C_4 \times C_4, Id, n), K_q)$ and $\mathsf{Hom}(G(P_k \times C_4, Id, n), K_q)$

$$1.4460096817417 \leq \lambda^c(3) \leq 1.4470681274660 \tag{9}$$

$$2.0343787307189 \leq \lambda^c(4) \leq 2.0652128520667 \tag{10}$$

As we can see the estimate from [2] for $\lambda^c(4)$ is within our bounds but their estimate for $\lambda^c(3)$ is well below the lower bound, even when their heuristic error estimate is taken into account. As mentioned in [2] this kind of "miss" by the series estimate could indicate a physically interesting structure in the set of 3-colourings.

# 5 Approximate Compression

For many applications the ring $F$ is the real numbers and one typically has only positive weights. If the aim is to compute only the maximum eigenvalue of $M$, as in Example 4, we can go further with our compression than in the previous section. First we here only need to care about the so called main part of the spectrum of $M$, i.e. the eigenvalues with eigenvector not orthogonal to $\mathbf{1}$, and of the main part we only need to preserve the maximum eigenvalue. We can now make use of one of the standard theorems of spectral graph theory, see e.g. [6].

**Theorem 5.1 (Interlacing of eigenvalues).** *Let $S$ be an $n \times m$ matrix such that $S^T S) = I$, $M$ a hermitian $n \times n$ matrix, and set $M' = S^T M S$. Let the eigenvalues of $M$ be $\lambda_1 \geq \lambda_2, \dots, \lambda_n$ and those of $M'$ be $\theta_1 \geq \theta_2, \dots, \geq \theta_m$. Then the eigenvalues of $M'$ interlace the eigenvalues of $M$, that is,*

$$\lambda_j \geq \theta_j \geq \lambda_{n-m+j}$$

**Corollary 5.2.** *Let $M$ be an $m \times m$ hermitian matrix and let $X = \{X_1, X_2, \dots\}$ be a partition of $\{1, \dots, m\}$. Define a matrix $B$ where $B_{i,j}$ is the average row sum in $M_{X_i,X_j}$. Then the eigenvalues of $B$ interlace those of $M$.*

*Proof.* Apply Theorem 5.1 with the matrix $S$ given by $S_{i,j} = |X_i|^{-\frac{1}{2}}$ if $j \in X_i$ and 0 elsewhere. □

For a partition which is not necessarily equitable we thus find

**Corollary 5.3.** *Let $\mathcal{X}$ be a partition of the rows and columns of $M$ then the maximum eigenvalue of $C(\mathcal{X})$ gives a lower bound on $\lambda_1(M)$.*

Given a partition $\mathcal{X}$ we can also define a matrix $D$ defined as in the previous corollary but using the maximum row sum rather than the average.

**Corollary 5.4.** *The maximum eigenvalue of $D$ gives an upper bound on $\lambda_1(M)$.*

For many choices of weighted graph $H$ it is the case that $Z(G, H)$ is either sub- or super-additive with respect to addition of edges and/or vertices to $G$. In this situation the corollaries of the interlacing theorem can be used to give us upper and lower bounds on the asymptotics of the maximum eigenvalues as $G$ becomes larger. These bounds can then in turn be used in combination with inequalities like 6.

In both Corollary 5.3 and 5.4 the choice of partition $\mathcal{X}$ will influence the value of the eigenvalue bound. How the partition should be chosen in order to get a good approximation will depend on the underlying graphs and weights and it is hard to say anything much more precise than that one should strive to get blocks $M_{X_i,X_j}$ with as closely concentrated row-sums as possible.

**Example 5.5.** In order to demonstrate the approximate bounds, and the influence of the choice of partition $\mathcal{X}$, we have computed these bounds for the transfer matrix for the number of 4-colourings on $P_{12} \times P_n$ and $C_{14} \times P_n$.

We have used two kinds of partition $\mathcal{X}$.

1. For the first type of partition we view each colouring as an integer written in base 4, for cycles we choose an arbitrary vertex to be the lowest digit. Next we sort the colourings as if they were integers.

   Given this sorted list of the colourings we partitioned the list into consequtive sublists of length $k$ and $k - 1$, with as few list of length $k - 1$ as possible.

2. As our second kind of partition we randomly partitioned the list of colours into parts of size $K$.

For these two partitions we next computed an upper bound for the largest eigenvalue of the transfer matrix for $C_{14}$ and a lower bound for that of $P_{12}$. For each graph and partitioning we choose $K$ as to give compressed matrices with side from 0.9 times the original side down to 0.1. For the random ordering we tried several random partitions.

In Figures 1 and 2 we have plotted the approximate eigenvalue divided by the correct eigenvalue. We find that the lower bounds tend to be more accurate than the upper bounds. In both cases the Integer encoding partition gives a noticeably better approximation than the random partitions. However, for the lower bound even the random partitions can be used to compress the matrix down to one tenth of its original side and still get a bound which is just one percent less than the correct value.

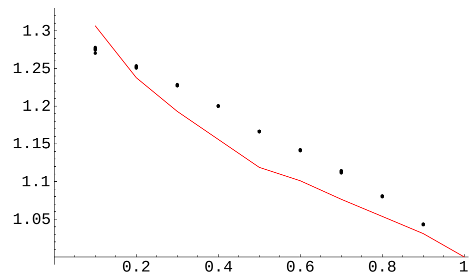


Figure 1: Approximate eigenvalues for $C_{14}$. Connected points are from the integer encoding. Clusters of isolated points are random partitions.

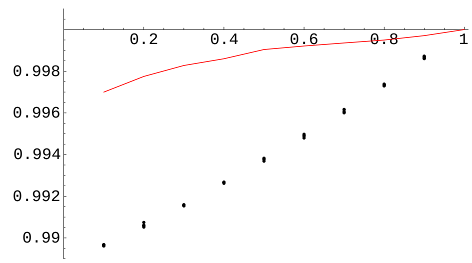


Figure 2: Approximate eigenvalues for $P_{12}$. Connected points are from the Integer encoding. Clusters of isolated points are random partitions.

# Acknowledgements

The computer resources needed for our computational work was provided by PDC, Stockholm, Sweden.

# References

[1] D. Babic et al. The matching polynomial of a polygraph. *Discrete Appl. Math.*, 15:11–24, 1986.

[2] A. V. Bakaev and V. I. Kabanovich. Series expansions for the $q$-colour problem on the square and cubic lattices. *J. Phys. A*, 27(20):6731–6739, 1994.

[3] Graham R. Brightwell and Peter Winkler. Graph homomorphisms and phase transitions. *J. Combin. Theory Ser. B*, 77(2):221–262, 1999.

[4] M. Ciucu. An improved upper bound for the 3-dimensional dimer problem. *Duke Math. J.*, 94(1):1–11, 1998.

[5] Shmuel Friedland and Uri N. Peled. Theory of computation of multidimensional entropy with an application to the monomer-dimer problem. *Adv. in Appl. Math.*, 34(3):486–522, 2005.

[6] Chris Godsil and Gordon Royle. *Algebraic graph theory*, volume 207 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.

[7] R. Häggkvist and P.H. Lundow. The Ising partition function for 2D grids with cyclic boundary: computation and analysis. *J. Statist. Phys.*, 108(3–4):429–457, 2002.

[8] Elliott H. Lieb. Residual entropy of square ice. *Phys. Rev.*, 162:162–172, 1967.

[9] Lovasz. hom. *Math.*, 4(3):486–522, 2005.

[10] Per Håkan Lundow. Compression of transfer matrices. *Discrete Math.*, 231(1-3):321–329, 2001. 17th British Combinatorial Conference (Canterbury, 1999).

[11] F. Y. Wu. The Potts model. *Rev. Modern Phys.*, 54(1):235–268, 1982.

# A Colouring eigenvalues for the square grid

| $k$ | $P_k$ | $C_k$ |
|---|---|---|
| 3 | 16.348469228349534294591 85 | 11 |
| 4 | 38.188748998197855775726 48 | 31.696938456699068589183 70 |
| 5 | 89.209728646509765238955 47 | 67.015148038438355607590 98 |
| 6 | 208.39809642539757206335 38 | 165.60082209445566724818 83 |
| 7 | 486.82915554135660005438 64 | 375.48040041528867979960 16 |
| 8 | 1137.2600584292242597979 68 | 892.24187534865773542127 83 |
| 9 | 2656.7036065885669863030 95 | 2064.5546065282126484470 34 |
| 10 | 6206.2098786660716404327 11 | 4849.5049433399230996427 84 |
| 11 | 14498.057634700719542682 93 | 11293.109165102436437817 10 |
| 12 | 33868.2836924334 | 26429.649584445686077498 08 |
| 13 | 79118.2289428612 | 61675.61597454731 |
| 14 | 184824.664073982 | 144167.2612085567 |
| 15 | 431760.883879445 | 336660.4085235824 |
| 16 | | 786626.0015010700 |

Table 2: Maximum eigenvalues for 4-colourings of the square grid

| $n$ | $P_n$ | $C_n$ |
|---|---|---|
| 2 | 13 | |
| 3 | 42.254746265138 | 32 |
| 4 | 137.34484848076 | 114.16796064692 |
| 5 | 446.42629917197 | 359.90932515034 |
| 6 | 1451.0662111085 | 1182.6934618883 |
| 7 | 4716.5527439510 | 3829.1466667249 |
| 8 | 15330.706253905 | 12464.383871815 |
| 9 | 49831.003080926 | 40492.334305935 |
| 10 | 161970.93773951 | 131643.39169572 |
| 11 | 526471.13343803 | 427861.13442804 |
| 12 | | 1390763.1270219 |

Table 3: Maximum eigenvalues for 5-colourings of the square grid

# B Colouring eigenvalues for the cubic lattice

| $n$ | $G = P_n$ | $G = C_n$ |
|---|---|---|
| 3 | 13.8072589673052 | 3 |
| 4 | 30.1109468160278 | 26.6214214843774 |
| 5 | 65.8601729387350 | 24.6080079665097 |
| 6 | 144.283083091960 | 123.199451464237 |
| 7 | 316.392676802175 | 148.219336059660 |
| 8 | 694.239161184508 | 582.950876259572 |
| 9 | 1523.97594483322 | 806.285660630712 |
| 10 | 3346.41981099416 | 2782.03759223168 |
| 11 | 7349.89922843146 | 4196.30975248900 |

Table 4: Maximum eigenvalues for 3-colourings of $G \times P_2$

| $n$ | $G = P_n$ | $G = C_n$ |
|---|---|---|
| 3 | 42.9509955498558 | 4.56155281280883 |
| 4 | 134.633390548866 | 114.548378741056 |
| 4 | 423.398960388624 | 103.444398290072 |
| 5 | 1333.80481197401 | 1091.43690942498 |
| 6 | 4206.08745616625 | 1449.29878537714 |
| 7 | | 10674.0945361673 |

Table 5: Maximum eigenvalues for 3-colourings of $G \times P_3$

| $n$ | $G = P_n$ | $G = C_n$ |
|---|---|---|
| 4 | 607.5008342289296 | 496.9033949197111 |
| 5 | 2751.292994653581 | 437.9397858090114 |
| 6 | 12483.36568754961 | 9768.207310946096 |

Table 6: Maximum eigenvalues for 3-colourings of $G \times P_4$

| $n$ | $G = P_n$ | $G = C_n$ |
|---|---|---|
| 5 | 17953.38896417563 | 1859.891162040439‘ |

Table 7: Maximum eigenvalues for 3-colourings of $G \times P_5$

| $n$ | $G = P_n$ | $G = C_n$ |
| --- | --- | --- |
| 2 | 3 | |
| 3 | 4.56155281280883 | 2 |
| 4 | 6.97196076839709 | 6.37228132326901 |
| 5 | 10.6828851212084 | 6 |
| 6 | 16.3920411989578 | 14.5064314940480 |
| 7 | 25.1740785316175 | 15.7833418763922 |
| 8 | 38.6831608665319 | 33.6767869577220 |
| 9 | 59.4651079147947 | 39.6505660120334 |
| 10 | 91.4379622705829 | 78.8188645918277 |
| 11 | 140.631735559805 | 97.5298788390751 |
| 12 | 216.327079158290 | 185.239857806635 |
| 13 | 332.808012753772 | 237.182998032027 |

Table 8: Maximum eigenvalues for 3-colourings of $G \times C_3$

| $n$ | $G = P_n$ | $G = C_n$ |
| --- | --- | --- |
| 2 | 26.62142148437744 | |
| 3 | 114.5483787410569 | |
| 4 | 496.9033949197111 | 420.477039628259 |
| 5 | 2163.237391033718 | 378.843114768611 |
| 6 | 9435.406059898469 | 7704.08921920854 |
| 7 | | 11291.7201866529 |
| 8 | | 144633.687249454 |

Table 9: Maximum eigenvalues for 3-colourings of $G \times C_4$

| $n$ | $G = P_n$ | $G = C_n$ |
| --- | --- | --- |
| 2 | 24.6080079665097 | |
| 3 | 103.444398290072 | |
| 4 | 437.939785809011 | |
| 5 | 1859.89116204043 | 408.155175807023 |
| 6 | 7912.06577168573 | 6610.84386549256 |

Table 10: Maximum eigenvalues for 3-colourings of $G \times C_5$

| $n$ | $G = P_n$ | $G = C_n$ |
| --- | --- | --- |
| 6 | | 599243.330687515 |

Table 11: Maximum eigenvalues for 3-colourings of $G \times C_6$