Cycle Double Covers and Spanning Minors II

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Abstract. In this paper we continue our investigations from [HM01] regarding spanning subgraphs which imply the existence of cycle double covers. We prove that if a cubic graph G has a spanning subgraph isomorphic to a subdivision of a bridgeless cubic graph on at most 10 vertices then G has a CDC. A notable result is thus that a cubic graph with a spanning Petersen minor has a CDC, a result also obtained by Goddyn [God88].

1. Introduction

A *cycle (or circuit) double cover*, or *CDC* for short, of a graph G is a collection of cycles in G, not necessarily distinct, such that any edge in G belongs to exactly two of the cycles. Here we use the currently standard graph theoretical definition of a cycle, a connected 2-regular graph, although in this subject it is often the case that the word cycle is used for a spanning subgraph with all vertex degrees even — by default we also use the word graph for simple graph.

The outstanding problem in the theory of cycle double covers, and by now one of the classic unsolved problems in graph theory, is the following

Conjecture 1.1. *Every* 2*-edge-connected graph has a cycle double cover.*

This conjecture has become known as the cycle double cover conjecture (CDCC) and is generally attributed to Seymour [Sey79] and Szekeres [Sze73]. The subject of graph embeddings is of course much older, and, as pointed out by Seymour, some forms of the conjecture may also have been present in earlier work by Tutte for instance. The conjecture was later strengthened by Celmins [Cel84] to

Conjecture 1.2. *There exists some integer $k \geq 5$ such that every* 2*-edge-connected graph has a k-cycle double cover.*

Here a CDC $\mathcal{C}$ is said to be a k-CDC if the cycles in $\mathcal{C}$ can be coloured with k colours in such a way that no two cycles of the same colour share an edge. Celmins further conjectures that conjecture 1.2 will hold with $k = 5$. For a more extensive survey of results concerning the cycle double cover conjecture see the introduction of [HM01]. However let us mention that the full version of conjecture 1.1 follows if it can be shown to hold for cubic graphs.

Key words and phrases. cycle double cover, Kotzig, frame, cubic graphs.

In [HM01] we introduced the terminology of frames as a way of studying cycle double covers. This has also been done independently by Goddyn in his thesis [God88]. We say that a graph H is a *frame* of a graph G if G has a spanning subgraph isomorphic to a subdivision of H and every component of the subdivision of H has an even number of inserted vertices. For cubic graphs this means that there is matching M in G such that $G \setminus M$ is a subdivision of H. We say that a graph H is a *good frame* if any graph with H as a frame has a CDC.

In [HM01] we considered several general classes of cubic graphs which are good frames, most notably the Kotzigian graphs. In this paper we will analyse the small cubic graphs and prove that all 2-connected cubic graphs on at most 10 vertices are good frames. We will show that in fact most of them give a 6-CDC. The most notable exception is the Petersen graph which, for our chosen ways of adding edges, will only give a 10-CDC. At the end we will also give a number of open problems.

2. Frames

From now on all graphs will be assumed to be cubic, i.e. 3-regular, unless otherwise specified. If a colouring of graph is mentioned it is meant to be an edge colouring of that graph. Proofs of statements in this section can be found in [HM01].

In order to be able to state our results in a convenient way we need a few definitions.

Definition 2.1. A multigraph H is said to be a *frame* of a graph G if G has a spanning subgraph $\widehat{H}$ such that,

(a) $\widehat{H}$ is isomorphic to a subdivision of H, and
(b) the number of vertices in each component $\widehat{H}_i$ of $\widehat{H}$ has the same parity as the number of vertices in the corresponding component H_i of H.

Or in other words a frame of a graph G is a spanning topological minor of G with an additional parity condition. A graph H is said to be a *good frame* if any cubic (2-edge-connected) graph with H as a frame has a CDC, and a k*-good frame* if any graph with H as a frame has a k-CDC.

Using this terminology the now classic folklore result that every hamiltonian cubic graph has a 3-CDC could be seen as saying that a 2-cycle is a 3-good frame. Similarly the result on hamiltonian paths by Tarsi [Tar86] means that K_2 is a good frame. Huck and Kochol's results on oddness 2 in [HK95] can be rephrased as saying that any collection of two K_3s together with an arbitrary number of C_4s is a 5-good frame. This also implies that K_2 is a 5-good frame.

Definition 2.2. Let $\hat{H}$ be a subdivision of cubic graph H. A vertex $v \in V(\hat{H})$, of degree 2, is said to *reside* on an edge e in H if v belongs to the path connecting the two vertices in $\hat{H}$ that correspond to the endpoints of e in H, obtained by subdividing the edge e.

Let H be a frame of G and $\hat{H}$ a subdivision of H spanning G. Then an edge e of $G \setminus \hat{H}$ is said to *string* a cycle C in H if both endpoints of e reside on edges in C. See Figure 1.

An edge e in $G \setminus \hat{H}$ is said to *connect* two disjoint cycles C_1 and C_2 in H if one endpoint of e resides on C_1 and the other resides on C_2.

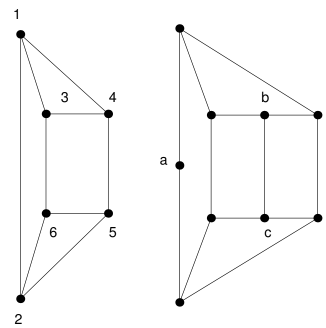

FIGURE 1. The vertex a resides on the edge $(1, 2)$. The edge (b, c) strings the cycle $3, 4, 5, 6$.

The following is a basic observation.

Proposition 2.3. [HM01] *Let H be a frame of a cubic graph G, and let M be a matching in G such that $G \setminus M$ is a subdivision of H. If there is a CDC $\mathcal{C}$ in H such that for every edge e in M there is a cycle $C_e \in \mathcal{C}$ for which e strings C_e, then G has a CDC, and if $\mathcal{C}$ is a k-CDC then G has a $2k$-CDC.*

The above proposition can be strengthened a bit to allow not only edges stringing the cycles of a CDC but also edges connecting two cycles in the CDC, although we need some conditions on how the edges connect these cycles. This is one such strengthening given in [HM01].

Proposition 2.4. [HM01] *Let H be a frame of a cubic graph G, and let M be a matching in G such that $G \setminus M$ is a subdivision of H. Let $\mathcal{C}$ be a CDC of H and $\mathcal{C}_2$ a set of pairwise disjoint cycles in $\mathcal{C}_2$.*

If every edge in M either strings a cycle in $\mathcal{C}$ or connects two cycles in $\mathcal{C}$ and every pair of cycles in $\mathcal{C}_2$ is connected by an even number of edges from M, then G has a CDC.

2.1. Kotzig Graphs. In order to make use of Proposition 2.3 to show that some graph H is a good frame we need to find a good CDC in H, preferably a CDC such that any pair on edges in H belongs to some cycle in the CDC. One large class of graphs which do have a CDC with this property is the class of kotzigian graphs, defined next.

Definition 2.5. Let G be a regular multigraph. A proper edge-colouring of G is said to be a *Kotzig colouring* of G if the union of the edges in any two colour classes forms a hamiltonian cycle in G.

A graph having a Kotzig colouring is said to be *kotzigian* or a *Kotzig graph*.

Kotzigian graphs were first introduced and studied by Anton Kotzig in [Kot58]. Kotzig first called kotzigian graphs *Hamiltonsche Graphen* (hamiltonian graphs) and later on *strongly hamiltonian* graphs. In recent years kotzigian graphs have been known as graphs with perfect one-factorisations, however in both this and the preceding paper we honour Kotzig by using the term kotzigian graphs.

In [Kot58] Kotzig studied bipartite regular graphs and found that in order for such graphs to be kotzigian their order had to be congruent to 2 modulo 4. After this introduction of the concept Kotzig went on to show, in [Kot62], that there exist two operations such that any cubic Kotzig graph can be constructed from an edge of multiplicity three by repeated use of these operations. Unfortunately these operations do not only give Kotzig graphs but rather Kotzig-coloured Kotzig graphs. Thus every Kotzig graph will be constructed with multiplicity equalling its number of Kotzig colourings, causing some headaches in an enumerative setting.

In [Kot64] Kotzig gives his first written exposition in English concerning kotzigian graphs, both citing his earlier results and also introducing new theorems. He first gives a new set of three planarity preserving operations and shows that any planar cubic Kotzig graph can be constructed using them. Next he shows that any Kotzig colouring of a cubic graph is equivalent to an euler tour compatible to a hamilton decomposition of a 4-regular graph and also gives a few results concerning the number of Kotzig colourings of a graph. Finally he shows that if $m-1$ is an odd prime then K_m is kotzigian and goes on to state the well known, and still open, conjecture that K_{2n} is kotzigian for all $n > 1$.

Much later Kotzig and Labelle gave a further study of the structure of kotzigian graphs in [KL78].

For kotzigian graphs Proposition 2.3 immediately gives us the following result.

Theorem 2.6. [HM01] *A kotzigian graph H is a 6-good frame.*

Proof. Since H is kotzigian it has a CDC consisting of the three 2-coloured hamiltonian cycles given by the Kotzig colouring and so the theorem follows from Proposition 2.3. □

Given a Kotzig graph there are a number of ways larger kotzigian graphs can be constructed. Let us recall one of them.

Proposition 2.7. [HM01] *Let G be a Kotzig graph and let e_1 and e_2 be two edges with different colours in some Kotzig colouring of G. The graphs G_1 and G_2 obtained by using the transformations in Figure 2 on e_1 and e_2 are Kotzig graphs.*

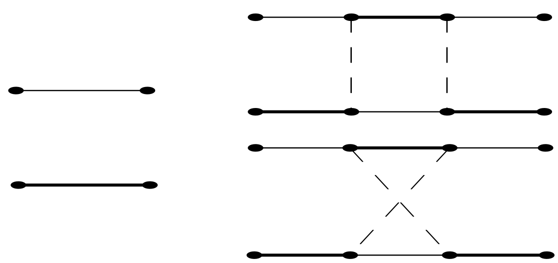

FIGURE 2. Adding a four-cycle

The *Möbius ladder*, M_k, on k spokes is the graph obtained from C_{2k} by adding edges between vertices i and $i+k$ for all $i \leq k$.

Corollary 2.8. *M_k is kotzigian for odd k, $k \geq 3$, and not kotzigian for even k, $k \geq 4$.*

Proof.

(1) Odd k. M_3 is isomorphic to $K_{3,3}$ and is a Kotzig graph, see Figure 3. For $k > 3$ the Möbius ladders can be constructed from M_3 by repeated use of Proposition 2.7.

(2) Even k. Assume that M_k is kotzigian for some even $k \geq 4$. Then we can use Proposition 2.7 in reverse to obtain a Kotzig colouring of M_{k-2}. Continuing in this way we find that $M_{k-4}, M_{k-6}, \ldots, M_4$ must be kotzigian. However if we continue this one step further, reducing M_4 to $M_2 \simeq K_4$, we find that even though K_4 is kotzigian its Kotzig colouring is not consistent with the conditions of Proposition 2.7. We thus have a contradiction and are done.

□

2.2. Graphs with a switchable CDC. In order to be kotzigian it is necessary for a graph to be 3-connected, but we have another useful kind of CDC which is possible for 2-connected graphs as well, so called switchable CDC's.

Definition 2.9. Let G be a 3-edge-colourable cubic graph. Assume that there is a 3-edge-colouring such that

(1) colours 1 and 2 together form a hamiltonian cycle,
(2) colours 1 and 3 together form a hamiltonian cycle,
(3) colours 2 and 3 form 2 disjoint 2-edge-coloured cycles, C_1 and C_2.

Assume further that if colours 2 and 3 are exchanged on the edges of C_2 we get a new 3-edge-colouring satisfying the three properties. Then each of these two CDC's is said to be a *switchable CDC*.

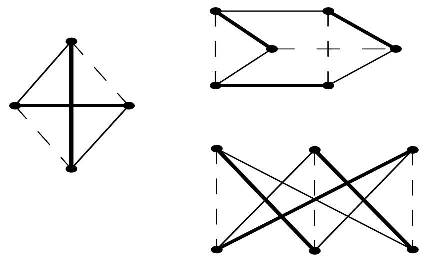

Figure 3. Kotzig colourings of all cubic graphs on four and six vertices.

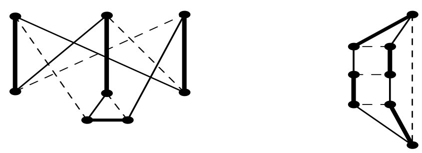

Figure 4. Kotzig colourings of the two Kotzig graphs on eight vertices.

Using Proposition 2.4 we get the following analogues of the results for Kotzig graphs.

Theorem 2.10. [HM01] *If A cubic graph H has a switchable CDC then H is 6-good frame.*

Theorem 2.11. [HM01] *Let G_1 and G_2 be two Kotzig graphs. Form a new graph H as follows. Let e_1 be an edge in G_1 and e_2 an edge in G_2. Delete e_1 and e_2 and add two new edges with one endpoint in G_1 and one endpoint in G_2 to obtain a new cubic graph H.*

The graph H has a switchable CDC.

3. Cubic Graphs on at most eight vertices as Frames

In this section we shall show that all 2-connected cubic graphs on at most eight vertices are good frames. For a complete list of all cubic graphs on at most ten vertices see [RW98], or [Roy] for a more computer friendly format.

Proposition 3.1. *All the graphs in Figures 3 and 4 have Kotzig colourings.*

Proof. For each of the above graphs a Kotzig colouring is shown in the figures. □

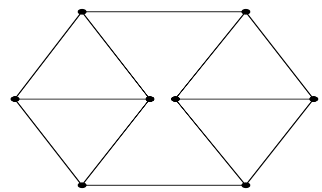

FIGURE 5

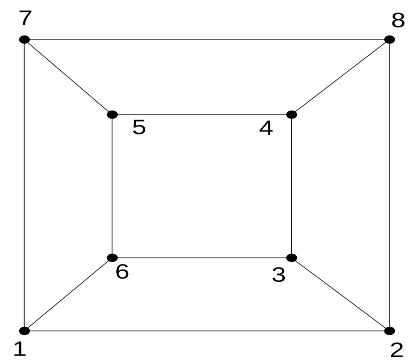

FIGURE 6. Q_3

In Figures 5, 6, and 9 we have the remaining, non-Kotzigian, cubic graphs on eight vertices: they are the cube, the Möbius ladder on eight vertices and one further graph.

Proposition 3.2. *The graph in Figure 5 is a 6-good frame.*

Proof. This graph can be obtained from two K_4's by using the operation in Theorem 2.11 and so has a switchable CDC. Hence it is by Theorem 2.10 a 6-good frame. □

Proposition 3.3. *The graph in Figure 6, the cube Q_3, is a 6-good frame.*

Proof. If we start with the CDC $\mathcal{C}$ given by the face cycles on a planar embedding of the cube we see that there are only two non-isomorphic ways to add edges to Q_3 not stringing a cycle in $\mathcal{C}$: either with endpoints residing on (1,2) and (3,4) or endpoints residing on (1,2) and (4,5). In Figure 7 and Figure 8 we show Kotzig colourings of the graph obtained after adding any of these two edges. So we assume that the only edges added string cycles in $\mathcal{C}$ and since $\mathcal{C}$ is a 3-CDC the result follows from Proposition 2.3. □

Proposition 3.4. *The graph in Figure 9, the Möbius ladder M_4, is an 8-good frame.*

Let G be a graph and M a matching such that $G \setminus M$ is a subdivision of M_4. If M contains exactly one edge of the kind shown in Figure 11, and

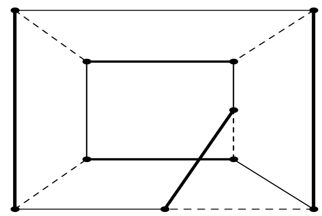

FIGURE 7

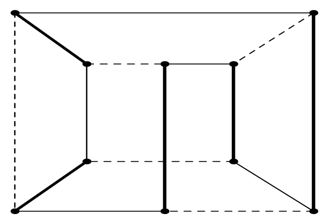

FIGURE 8

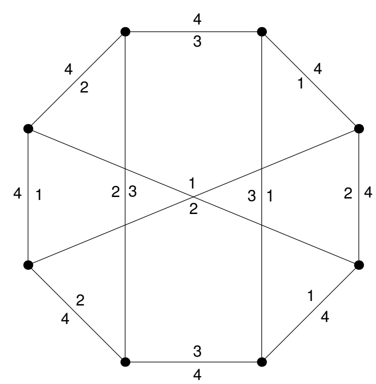

FIGURE 9. The 4-spoke Möbius ladder M_4.

none of the kind shown in Figure 10, then G has an 8-CDC, otherwise G has a 6-CDC.

Proof. First consider the 3-CDC given by colouring the edges on the outer hamiltonian cycle in Figure 9 alternatingly red/green and the inner edges blue. If all edges span a cycle of the CDC we are done by Proposition 2.3.

There are only two non-isomorphic ways to add an edge not spanning a cycle of the 3-CDC. In Figure 10 we show a Kotzig colouring of the first of the two graphs so obtained.

In case we have exactly one edge of the kind shown in Figure 11, and no edge like that in Figure 10, we use the 4-CDC shown in Figure 9 to construct an 8-CDC using Proposition 2.3. If there is more than one edge

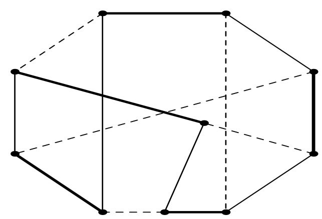

FIGURE 10

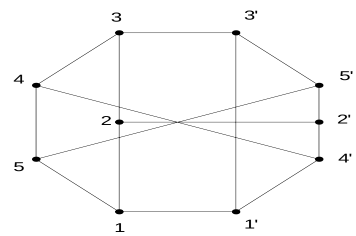

FIGURE 11. The Petersen graph.

of this kind we observe that every graph obtained by adding two such edges is isomorphic to one of the graphs that will be proved to be a 6-good frame in Proposition 4.7. □

4. Cubic Graphs on ten vertices as Frames

We will now show that all 2-connected cubic graphs on ten vertices are good frames. We first list all the graphs which are kotzigian or have switchable CDCs and then go on to treat the remaining graphs individually.

Proposition 4.1. *All the graphs in Figure 12 have Kotzig colourings.*

Proof. For each of the above graphs a Kotzig colouring is shown in the corresponding figure. □

Proposition 4.2. *The graphs in Figure 13 have switchable CDCs.*

Proof. Each graph has a single 2-edge cut and can be constructed from smaller Kotzig graphs as described in Theorem 2.11. □

Our next graph has two 2-cuts and can thus neither be kotzigian nor have a switchable CDC. However due to its nice structure it can be handled using a variation on switchable CDCs.

Proposition 4.3. *The graph H in Figure 14 is a 6-good frame.*

Proof. Let G be a graph with H as a frame. Consider the CDC $\mathcal{C}$ indicated by the 3-colouring of H shown in 14. If all edges added to H string cycles in $\mathcal{C}$ we are done and so we assume that this is not the case.

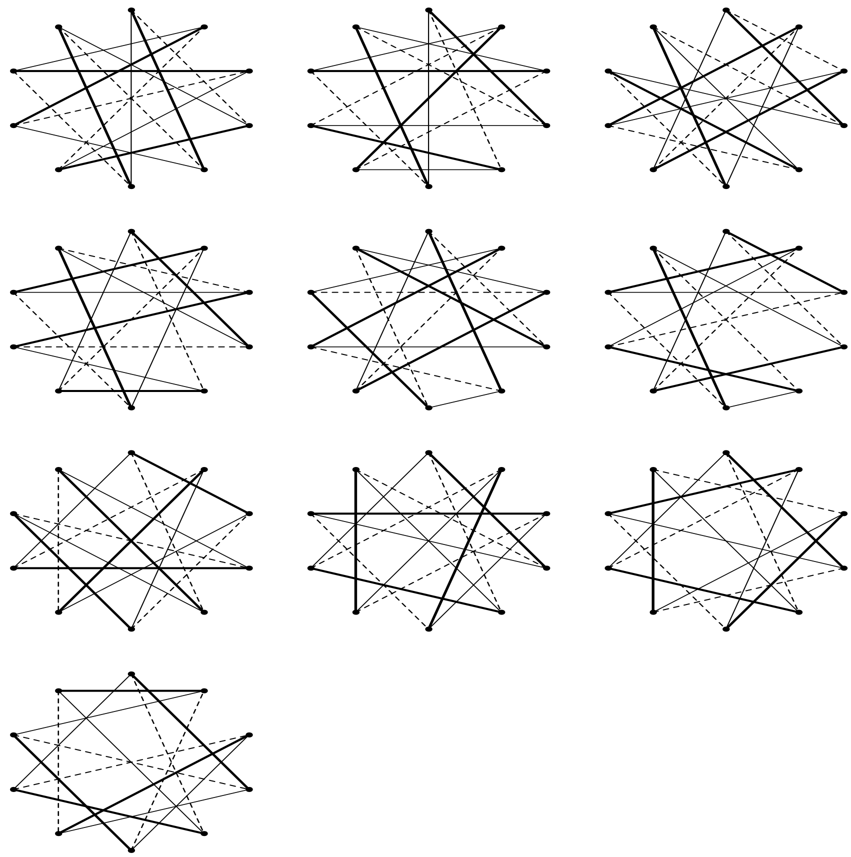

FIGURE 12. Kotzig colourings of all kotzigian cubic graphs on ten vertices.

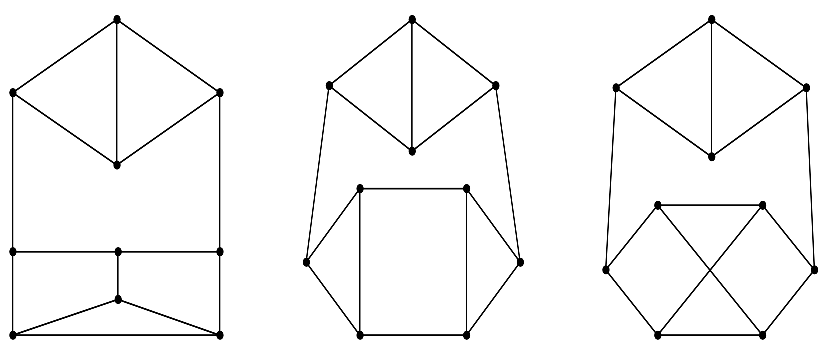

FIGURE 13. The graphs on ten vertices with switchable CDCs.

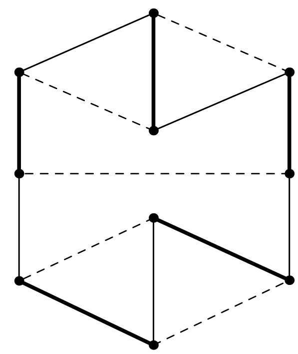

FIGURE 14

We will now look at the different ways of adding an edge to H which does not string a cycle in $\mathcal{C}$. We get two cases.

(1) Assume that there is an edge in G with one endpoint residing on the horizontal edge in H and one endpoint on an edge not adjacent to the horizontal edge.

There is only one non-isomorphic way to add an edge in this way and the resulting graph has a unique 2-edge cut. Cutting the graph along this cut allows us to reconstruct the graph using the method of Theorem 2.11 and the graph thus has a switchable CDC and we are done.

That the graphs obtained after cutting along the 2-cut are kotzigian is easily seen since none of the 3-connected non-Kotzig graphs on eight vertices has a triangle.

(2) Assume that there is no edge as in case (1) but there are edges connecting the two 4-cycles in H. If necessary we can switch the colours used on one of the 4-cycles and thereafter choose one of the edges to string a hamiltonian cycle rather than connect the 4-cycles thus making sure that there is an even number of edges connecting the two 4-cycles and we are done by Proposition 2.4.

□

We next go on to the graphs obtained by replacing a vertex in the cube or the Möbius strip by a triangle. Although neither of these graphs is kotzigian we quickly end up with Kotzig graphs by adding edges not stringing cycles of the given CDCs.

Proposition 4.4. *The graph H in Figure 15 is a 6-good frame.*

Proof. Let G be a graph with H as a frame. Unless there is an edge in G with an endpoint residing on the left sloping edge in the triangle in H the graph G will have the cube Q_3 as a frame and thus a 6-CDC by our previous results. We thus assume that there is such an edge, e.

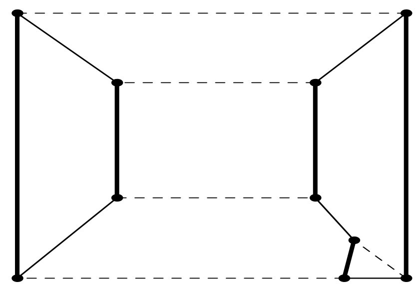

Figure 15

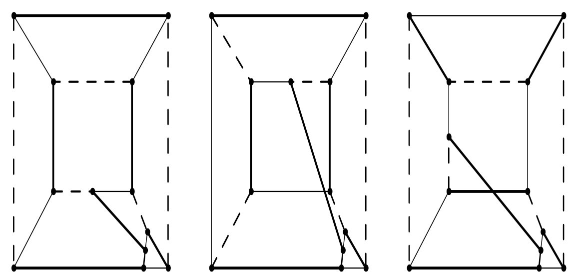

Figure 16

If we consider the CDC given by the 3-edge-colouring shown in Figure 15 we see that there are only three non-isomorphic ways to add e if it is not stringing a cycle of the CDC. In Figure 16 we show the three graphs so obtained and for each we display a Kotzig colouring. $\square$

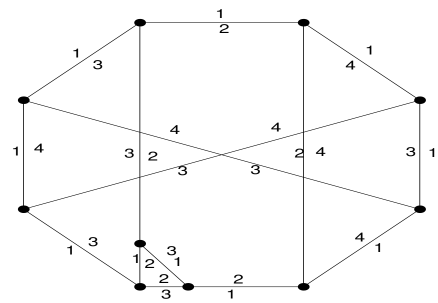

FIGURE 17

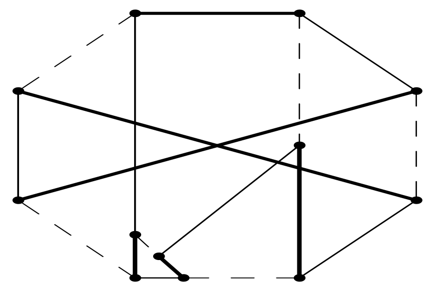

FIGURE 18

Proposition 4.5. *The graph H in Figure 17 is an 8-good frame.*

Let G be a graph and M a matching such that $G \setminus M$ is a subdivision of H. If M contains exactly one edge e such that $(G \setminus M) \cup e$ is isomorphic to the graph obtained by expanding vertex 1 in Figure 11 into a triangle and no edge of the type shown in Figure 18 then G has an 8-CDC, otherwise G has a 6-CDC.

Proof. Let G be a graph with H as a frame. Unless there is an edge of G with one endpoint residing on the right sloping edge of the triangle G has the Möbius ladder M_4 as a frame and we are done by Proposition 3.4.

Let us assume that there is such an edge and consider the CDC displayed in Figure 17. We see that there is only one way of adding an edge not stringing a cycle of the CDC and in Figure 18 we show a Kotzig colouring of the graph obtained. $\square$

Our last two graphs are the Petersen graph and the 5-prism ($K_2 \square C_5$). They are structurally similar in that they consist of two 5-cycles joined by a matching but as we shall see the prism is a slightly better frame.

Proposition 4.6. *The graph $K_2 \square C_5$, Figure 19, is a 6-good frame.*

Proof. Let G be a graph and M a matching in G such that $H = G \setminus M$ is a subdivision of $K_2 \square C_5$.

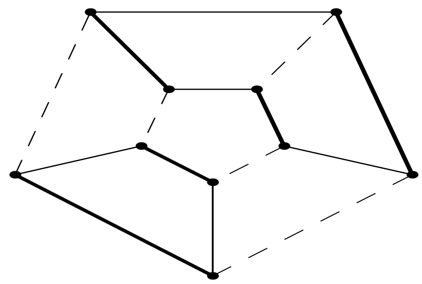

FIGURE 19. $K_2 \square C_5$

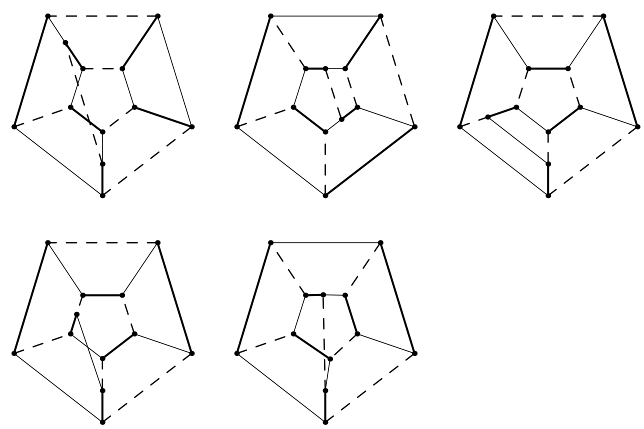

FIGURE 20. The top left graph has a switchable CDC; the remaining graphs have a Kotzig colouring.

If M contains an edge e such that $H \cup e$ is isomorphic to any of the graphs in Figure 20 we have either a Kotzig colouring of $H \cup e$ or a colouring giving a switchable CDC and we are done. We thus assume that no such edge is present.

If there is an edge such that $H \cup e$ is isomorphic to the graph in Figure 21 we find that there are four different ways of adding edges not spanning the CDC shown in the Figure, and in Figure 22 we show Kotzig colourings of the four graphs so obtained. So we assume that no such edge e is present.

If there is an edge such that $H \cup e$ is isomorphic to the graph in Figure 23 we find that there are three different ways of adding edges not stringing the CDC shown in the figure, and in Figure 24 we show Kotzig colourings of the three graphs so obtained. So we assume that no such edge e is present.

Finally if there are no edges of the previous kinds then every edge in M must string a cycle in the CDC shown in Figure 19 and we are done. $\square$

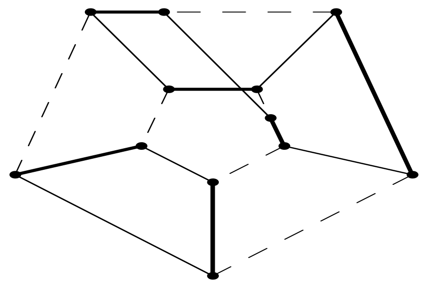

Figure 21

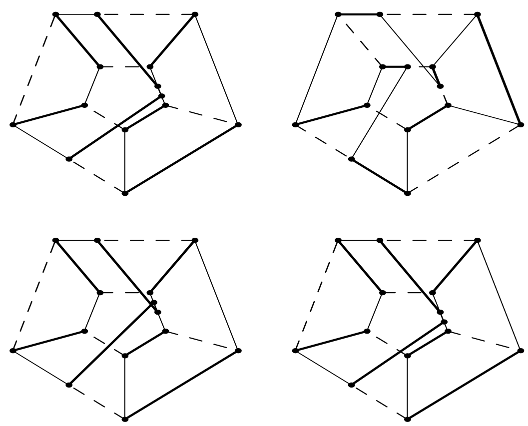

Figure 22

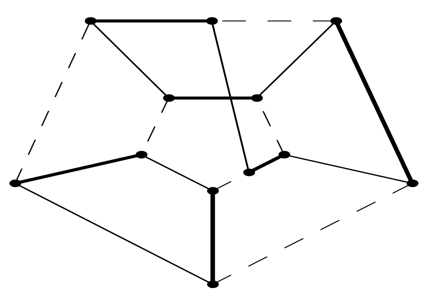

Figure 23

Proposition 4.7. *A cubic graph, constructed from the Petersen graph by subdividing two non-adjacent edges and connecting the two new vertices by an edge, is a 6-good frame.*

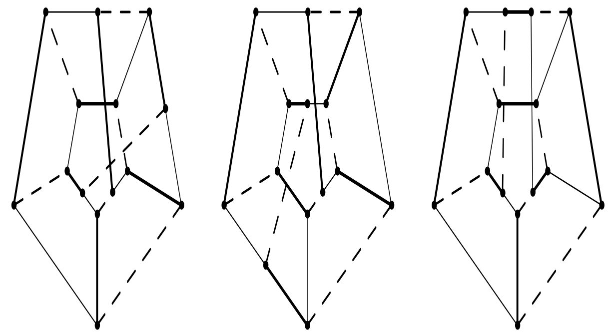

Figure 24

Proof. There are only two non-isomorphic ways to add an edge of the kind mentioned in the proposition; the graphs obtained are shown in Figure 25 and Figure 26. We now examine the two cases.

(1) It is easily checked that the given 3-colouring of the graph in Figure 25 gives a switchable CDC. Exchanging the colour on the outer 6-cycle gives two new hamiltonian cycles. If there is an edge like the one added to obtain this graph we are done.
(2) If we consider the CDC given by the 3-edge-colouring in Figure 26 we see that there are at most eight non-isomorphic ways to add an edge not spanning a cycle in the CDC.
    (a) Adding an edge with endpoints residing on edges (0,5) and (6,11). In figure 27 we show a Kotzig colouring of the graph obtained.
    (b) Adding an edge with endpoints residing on edges (0,5) and (4,9). In Figure 28 we show a Kotzig colouring of the graph obtained.
    (c) Adding an edge with endpoints residing on edges (0,5) and (3,8). In Figure 29 we show a Kotzig colouring of the graph obtained.
    (d) Adding an edge with endpoints residing on edges (5,7) and (3,4). The edge added here brings us back to case (1).
    (e) Adding an edge with endpoints residing on edges (5,7) and (9,11). This graph is isomorphic to the one obtained in case (h), where a Kotzig colouring is displayed.
    (f) Adding an edge with endpoints residing on edges (1,10) and (3,8). The edge added here brings us back to case (1).
    (g) Adding an edge with endpoints residing on edges (1,10) and (6,11). This graph is isomorphic to the one obtained in case (i).
    (h) Adding an edge with endpoints residing on edges (0,10) and (3,8). In Figure 30 we show a Kotzig colouring of the graph obtained.

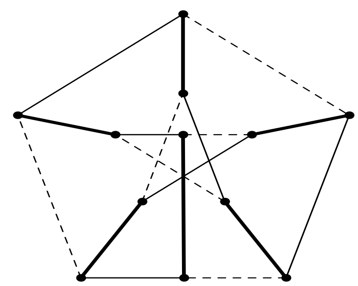

FIGURE 25

(i) Adding an edge with endpoints residing on edges (0,10) and (9,11). This graph has a switchable CDC. In figure 31 we show a 3 edge colouring of this graph, such that the colours on the outer 6-cycle can be interchanged to obtained a new CDC with the desired properties.

□

Proposition 4.8.

(i) *Let G be a cubic graph and M a matching in G such that $G \setminus M$ is a subdivision of the Petersen graph P. If M contains an edge with endpoints residing on non-adjacent edges in P then G has a 6-CDC.*
(ii) *Let G be a cubic graph and M a matching in G such that $G \setminus M$ is a subdivision of P. If all edges of M have endpoints on adjacent edges in P and at least one edge in P is not subdivided then G has an 8-CDC.*
(iii) *The Petersen graph P is a 10-good frame.*

Proof. Part (i) follows from Proposition 4.7. For part (ii) we delete the undivided edge and observe that the Petersen graph is edge-transitive. Thus we can find a subdivision of the Möbius-ladder spanning $G \setminus e$ and apply Proposition 3.4. Finally for part (iii) we can assume that every edge in M has endpoints residing on adjacent edges in the Petersen graph, thus every edge in M must string a cycle in the 5-CDC of the Petersen graph, consisting of 5 five cycles, and we are done by Proposition 2.3. □

5. Discussion, Corollaries, and Conjectures.

Our first Theorem just sums up the results in the last two sections.

Theorem 5.1. *A cubic graph with a 2-connected cubic frame on at most ten vertices has a CDC.*

Corollary 5.2. *If a 2-connected cubic graph G does not have a CDC then $G \setminus M$ is not 2-connected for any $(\frac{n}{2} - k)$-matching M, $k = 1, \dots, 5$, M in G.*

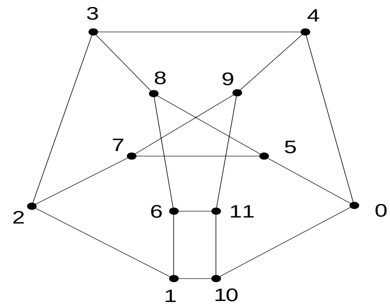

Figure 26

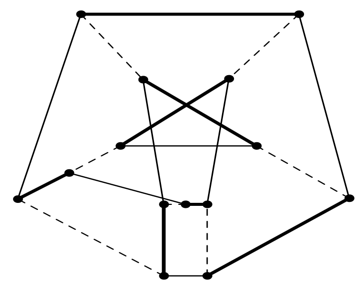

Figure 27

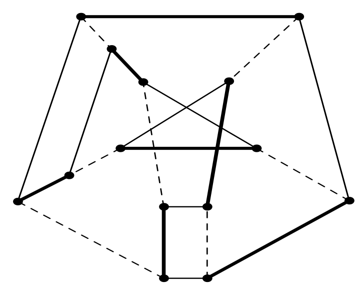

Figure 28

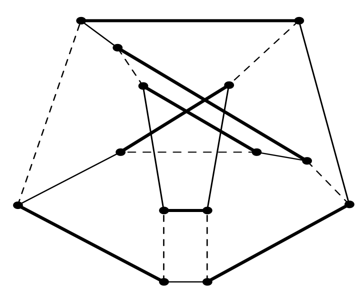

Figure 29

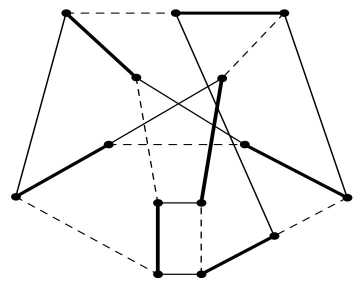

FIGURE 30

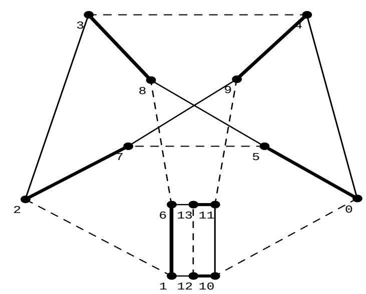

FIGURE 31

Proof. The graph $G \setminus M$ will have $2k$ vertices of degree three and so has a cubic frame on $2k$ vertices. The corollary now follows from Theorem 5.1. $\square$

We can also derive the following Corollary

Corollary 5.3. *A cubic graph G on n vertices without a CDC has no cycles of length $n-k$ for $k = 1, \ldots, 3$.*

However we can draw an even stronger conclusion. In [HM00] [Huc01] it was proven that a graph with a 2-factor with at most 4 odd cycles has a CDC. Although it was not noted in there this implies that a graph with a cycle of length $l, l \in \{n, \ldots, n-4\}$, has a CDC. This can be seen by blowing up the vertices outside the cycle to triangles and considering the 2-factor given by the triangles and the long cycle.

For most of the small graphs analysed we have shown that they are 6-good frames. However if one examines the proofs showing that the other graphs are k-good for larger k we see that in order not to obtain a 6-CDC one must add edges in a very precise way. This leads us to believe that a more careful, and probably much longer, analysis would give the following:

Conjecture 5.4. *All two-connected cubic graphs on at most ten vertices are 6-good frames.*

Of particular interest in this respect is of course the Petersen graph.

Considering the result in section 2 showing that the Möbius ladders M_k are kotzigian for odd k and not for even k the following would seem likely:

Conjecture 5.5. *There is a j such that the Möbius ladders M_k are j-good frames for all even k.*

Similarly observing that we have shown that all small prisms are 6-good frames one could ask:

Problem 5.6. *Is there a k such that all prisms $K_2 \square C_n$ are k-good frames?*

Another direction for generalisation is prototyped by the following result, which can be proven using the methods of [HM01],

Proposition 5.7. *Let H be the disjoint union of any number of even cycles and Q_3. Then the graph H is a 6-good frame.*

Here one could probably prove a corresponding statement for all other 2-connected cubic graphs on at most 10 vertices. In fact the Petersen graph seems especially amenable.

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