Cycle Double Covers and Spanning Minors I

Roland Häggkvist and Klas Markström

Department of Mathematics, UmeåUniversity , SE-901 87 Umeå, Sweden
E-mail: Roland.Haggkvist@math.umu.se
E-mail: Klas.Markstrom@math.umu.se

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Define a graph to be a Kotzig graph if it is m-regular and has an m-edge colouring in which each pair of colours form a Hamiltonian cycle. We show that every cubic graph with spanning subgraph consisting of a subdivision of a Kotzig graph together with even cycles has a cycle double cover, in fact a 6-CDC. We prove this for two other families of graphs similar to Kotzig graphs as well.

In particular, let F be a 2-factor in a cubic graph G and denote by G_F the pseudograph obtained by contracting each component in F. We show that if there exist a cycle in G_F through all vertices of odd degree, then G has a CDC.

We conjecture that every 3-connected cubic graph contains a spanning subgraph homeomorphic to a Kotzig graph.

In a sequel we show that every cubic graph with a spanning homeomorph of a 2-connected cubic graph on at most 10 vertices has a CDC.

1. INTRODUCTION

A *cycle (or circuit) double cover* of a graph G is a collection of cycles in G, not necessarily distinct, such that any edge in G belongs to exactly two of the cycles. Here we use the currently standard graph theoretical definition of a cycle, a connected 2-regular graph, although in this subject it is often the case that the word cycle is used for a spanning subgraph with all vertex degrees even.

One of the original motivations for studying *cycle double covers*, from now on abbreviated CDCs, of a graph G is the fact that they correspond to nice embeddings of the graph in some surface, in which case the cycles in the CDC play the role of the face cycles in a planar embedding of a planar 2-connected graph. Here "nice" means that every face of the embedding is bounded by a cycle, an embedding with this property is called a *closed*

2-cell embedding. The outstanding problem in the theory of cycle double covers, and by now one of the classic unsolved problems in graph theory, is the following

Conjecture 1.1. Every 2-edge-connected graph has a cycle double cover.

This conjecture has become known as the cycle double cover conjecture (CDCC) and is generally attributed to Seymour [24] and Szekeres [25]. The subject of graph embeddings is of course much older, and, as pointed out by Seymour, some forms of the conjecture may also have been present in earlier work by Tutte for instance. The conjecture was later strengthened by Celmins [4] to

Conjecture 1.2. There exists some integer $k \geq 5$ such that every 2-edge-connected graph has a k-cycle double cover.

where a CDC $\mathcal{C}$ is said to be a k-CDC if the cycles in $\mathcal{C}$ can be coloured with k colours in such a way that no two cycles of the same colour share an edge.

In more recent years work on the embedding version of the conjectures have been done by Zha, proving that graphs embeddable in an orientable surface of genus 1 or 2 [29], or a nonorientable surface of genus at most 5 [30] has closed 2-cell embeddings. In [21] Zha and Robertson show that graphs without Möbius-ladder minors have closed 2-cell embeddings as well.

Yet another train of investigation comes from the theory of integer flows. For a thorough treatment of this see [32].

Despite much work both the above conjectures remain open. For the contents of this paper the most relevant directions of study are those which give conditions implying that a graph has a CDC, and those which give properties of a possible smallest counterexample. For a good introductory survey to the topic of cycle double covers see [13].

One of the basic results in this is that the CDCC for cubic graphs implies the CDCC for all graphs and consequently work on the CDCC has been concentrated on cubic graphs, the current paper being no exception.

However, let us first mention two conditions which ensure that a general bridgeless graph has a CDC. The first such condition is that it contains a spanning eulerian subgraph (such graphs are called supereulerian and are much studied in their own right). Every graph containing a pair of edge disjoint spanning trees (as for instance every 4-edge-connected graph) is supereulerian, and consequently every 4-edge-connected graph has a CDC. This fact was pointed out by Jaeger in his work on the so called 8-flow theorem. Indeed Jaeger has shown that every bridgeless graph without 3-edge-cuts has a CDC. A general reference is still [13] where further references can be found. The next result along these lines is less obvious to use and harder to find in the literature. It says that a bridgeless graph admits a CDC if it contains a spanning subgraph where every component is eulerian and has an even number of edges in its boundary (one corollary would be that any bridgeless bipartite graph with a 2-factor has a CDC if all degrees are odd).

This condition applied to a cubic graph says that a cubic graph has a CDC if it is 3-edge-colourable. In order to obtain a CDC (in fact a 3-CDC) it suffices to consider the three 2-factors of G given by each pair of colour classes among the edges. From this it follows immediately that the following classes of cubic graphs have CDCs: hamiltonian graphs, bipartite graphs and, by default, graphs with a 2-factor with only even cycles. The last class of graphs make the following definition interesting

DEFINITION 1.1. The *oddness* of a graph G, denoted $o(G)$ is the smallest number of odd cycles in any 2-factor of G.

In [12] Huck and Kochol proved that a cubic bridgeless graph with $o(G) = 2$ has a 5-CDC and more recently Häggkvist & McGuinness [8] and independently Huck, [11] proved that a cubic graph with $o(G) = 4$ has a CDC. As a corollary of the result on oddness 2 a cubic bridgeless graph with a hamiltonian path has a 5-CDC, thus improving Tarsi's seminal theorem in [26], that every cubic bridgeless graph with a hamiltonian path has a CDC (Tarsi obtained a 6-CDC).

In [20], see [27] for complete reference, Robertson, Seymour, and Thomas proved Tutte's conjecture that every 2-connected cubic graph with no Petersen minor is 3-edge-colourable and thus also has a 3-CDC. Along the same line of investigation Huck [10], using an unpublished result from [22], has proved that a graph not having a Petersen-minor has a 5-CDC which can be constructed in polynomial time.

In the other direction, results on the properties of minimal counter-examples, basic results [13] are that a minimal counterexample must be 3-edge-connected, cyclically-4-edge-connected (meaning that at least four edges must deleted in order to get two components containing cycles), not be 3-edge-colourable, and have girth at least 4. Graphs with these properties has become known as *snarks* and the study of their construction and properties has become an industry of it's own, see for example [14] and [3] and their references. Note in particular that in the cited paper Kochol constructs snarks of arbitrary large given girth. All snarks on at most 28 vertices has been constructed by Gunnar Brinkmann and are available at Gordon Royles homepage [23].

The requirement on the girth was first improved by McGuinness [19] and Goddyn [6] to 8 and much later by Huck [10] to 12. Huck also shows that any graph not having a 5-CDC must have girth at least 10.

In the late seventies and early eighties the cycle double conjecture was studied heavily, at for instance the department of Combinatorics and Optimisation, University of Waterloo, Ontario (Canada?) and as a result of this activity a number of doctoral dissertations and masters thesis emerged on the subject, usually in conjunction with some study of various related flow conjectures. Few of the results from these dissertations have been published separately, but the rare copies of the Thesae that we via our interlibrary service have got hold of have been very informative. The thesis by Luis Goddyn in particular has reached cult status. We have also found the Master's Thesis by Sean McGuinness and chapter 4 of Uldis Celmins Doctoral Thesis well worth acquiring. Fortunately we did not start this project by looking at those hard to get masterpieces, otherwise we probably would have given up much earlier, but rather we plunged right in by trying to answer the following natural question: Is it true that any cubic graph with a spanning homeomorph (subdivision) of the Petersen graph has a CDC? Those favoured by a copy of Goddyn's thesis find the answer there. Yes, it has a 10-CDC. In the current paper or its sequel we shall prove that every cubic graph which contains a spanning homeomorph of a(ny) 2-connected cubic multigraph on at most 10 vertices has a CDC.

In order to attack this question we unwittingly take the path already trodden by Goddyn, but we shall add some twists of our own. It turns out that any cubic graph that contains a spanning homeomorph of a cubic graph with a 3-edge-colouring where each pair of colours give a hamilton cycle (such graphs, be they cubic or no, we shall call kotzigian) has a CDC. Thus spake Goddyn and so say we. We say more though. Indeed, it turns out that any cubic graph which contains a spanning subgraph with one component homeomorphic to a kotzigian graph and the other components all even cycles has a CDC. Does anyone know of a 3-connected cubic graph without a spanning homeomorph of some kotzigian graph? We do not. It is true that any bridgeless cubic graph has a spanning subgraph consisting of a theta-graph and a collection of cycles,. However one can not guarantee that the cycles are even [7].

An extension of the kotzigian cubic graphs that we shall call iterated kotzigian graphs is the following: 1. Any kotzigian cubic graph with a kotzig colouring, one colour class of which is *blue* is iterated kotzigian. 2. An iterated kotzigian graph is obtained from a smaller iterated kotzigian graph H by inserting an iterated kotzig graph G with specified blue edges into one blue edge of H. (insert= pick a blue edge uv in G, and blue edge wx in H, join u to w, and v to x by a blue edge and delete uv and wx.)

Goddyn says that any cubic graph with a spanning homeomorph which is an iterated Kotzig graph has a CDC. We agree and top this by saying that any cubic graph which has a spanning subgraph where one component is homeomorphic to an iterated Kotzig graph and the remaining components all are even cycles has a CDC.

Yet another extension of kotzigian graphs are the graphs with a switchable CDC. We omit the precise definition in this introduction. Once more both we and Goddyn have found them to be good frames, but we can extend their use by allowing further even cycle components as above.

Based on the above we conjecture that a cubic bridgeless graph containing a spanning homeomorph H^* of a graph H such that

a) every component of H^* is of even order, and
b) every such component is kotzigian (or iterated kotzigian or has a switchable CDC) can be *proved* to have a CDC.

No doubt we shall return to this question elsewhere.

2. FRAMES

From now on all graphs will be assumed to be cubic, i.e 3-regular, unless otherwise specified. If a colouring of a graph is mentioned it is meant to be an edge colouring of that graph.

In order to be able to state our result in a convenient way we need a few definitions.

DEFINITION 2.1. A bridgeless cubic graph H is said to be a *frame* of a graph G if G has a spanning subgraph $\widehat{H}$ such that,

(a) $\widehat{H}$ is isomorphic to a subdivision of H, and
(b) the number of vertices in each component $\widehat{H}_i$ of $\widehat{H}$ has the same parity as the number of vertices in the corresponding component H_i of H.

Or in other words a frame of a graph G is a spanning topological minor of G with an additional parity condition. A graph H is said to be a *good frame* if any cubic (2-edge-connected) graph with H as a frame has a CDC, and a *k-good frame* if any graph with H as a frame has a k-CDC.

DEFINITION 2.2. Let $\hat{H}$ be a subdivision of cubic graph H. A vertex $v \in V(\hat{H})$, of degree 2, is said to *reside* on an edge e in H if v belongs to a path connecting the two vertices in $\hat{H}$ that correspond to the endpoints of e in H, obtained by subdividing the edge e.

Let H be a frame of G and $\hat{H}$ a subdivision of H spanning G. Then an edge e of G is said to *string* a cycle C in H if both endpoints of e resides on edges in H. See Fig 1.

An edge e in G is said to *connect* two disjoint cycles C_1 and C_2 in H if one endpoint of e resides on C_1 and the other resides on C_2.

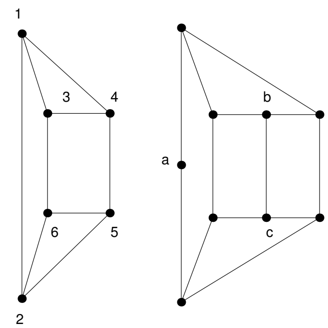

FIG. 1. The vertex a resides on the edge $(1, 2)$. The edge (b, c) strings the cycle $3, 4, 5, 6$.

Note that since every edge belongs to two cycles in a CDC an edge in G can sometimes be chosen either to string a cycle C_1 or to string another cycle C_2 sharing two edges with C_1.

The following is a basic observation,

PROPOSITION 2.1. *Let G be a cubic graph, H a frame of G and let M be a matching in G such that $G \setminus M$ is a subdivision of H. If there is a CDC $\mathcal{C}$ in H such that for every edge e in M there is a cycle C_e such that e strings C_e, then G has a CDC and if $\mathcal{C}$ has k cycles then G has a $2k$-CDC.*

Proof. We will prove the Proposition for the case when all the edges in M string the same cycle. The theorem then follows by induction on the number of cycles strung by the edges in M.

Let $\hat{C}$ be the cycle in the subdivision of H corresponding to the cycle in H strung by the edges in M. We will now construct a set of cycles $\tilde{C}$ in G such that every edge in $\hat{C}$ is covered once by these cycles and every edge in M is covered twice.

Colour every edge in M red and colour the paths along $\hat{C}$ green and blue, changing colour at the endpoints of the edges in M. Since there is an even number of endpoints this will be possible. Now the cycles formed by the red edges and the green paths, and the red edges and the blue paths give us the set of cycles $\tilde{C}$.

The set $\tilde{C}$ together with the cycles in G corresponding to the cycles in $\mathcal{C} \setminus \hat{C}$ gives us a CDC of G.

Since every cycle in $\mathcal{C}$ gives rise to two collections of disjoint cycles in the CDC of G, G has a $2k$-CDC. ▮

The above proposition can be strengthened a bit to allow not only edges stringing the cycles of a CDC but also edges connecting two cycles in the

CDC, although we need some conditions on how the edges connect these cycles.

PROPOSITION 2.2. *Let G be a cubic graph, H a frame of G and let M be a matching in G such that $G \setminus M$ is a subdivision of H. Let $\mathcal{C}$ be a CDC of H and $\mathcal{C}_2$ a set of pairwise disjoint cycles in $\mathcal{C}$.*

If every edge in M either strings a cycle in $\mathcal{C}$ or connects two cycles in $\mathcal{C}$ and every pair of cycles in $\mathcal{C}_2$ is connected by an even number of edges from M, then G has a CDC.

Proof. The edges whose endpoints do not reside on any cycle in C_2 can be dealt with as in the proof of Proposition 2.1 and so we will move on to the edges stringing or connecting cycles in C_2.

Colour each edge in M red and colour the paths between their endpoints alternatingly green and blue. Since there is an even number of endpoints residing on each cycle and the cycles are disjoint this can be done. The cycles formed by each pair of colours now cover each edge in M twice and the edges on C_2 once and together with the cycles corresponding to $\mathcal{C} \setminus C_2$ form a CDC of G ▌

We note that this theorem can in fact be generalised by assuming that there are several sets of cycles C_2, C_3 ... like the set C_2 in the theorem and allowing an even number of edges connecting the cycles within each set.

As an application of Proposition 2.2 we can show

THEOREM 2.1. *Let G be cubic a graph, H a frame of G and let M be a matching in G such that $G \setminus M$ is a subdivision of H. If H is 3-edge-colourable and every pair of edges in H is connected by an even number of edges in G, then G has a CDC.*

Proof. We will consider a CDC $\mathcal{C}$ of H given by an edge 3-colouring and in three steps construct a CDC of G. The fact that all edge pairs are connected by an even number of edges will allow us to use 2.2.

1. Consider the 2-factor of H induced by colours I and II. Construct a new graph G_1 from H by adding all edges in G connecting edges of colours (Red,Red), (Red,Green), or (Green,Green). From $\mathcal{C}$ we can construct a CDC $\mathcal{C}_\infty$ according to 2.2. In this new CDC the cycles corresponding to colours (Green,Blue), and (Red,Blue) still remain.
2. Now consider the set of cycles in G_1 corresponding to the cycles given by colours (Green,blue) in H. We construct G_2 by adding the edges in G connecting edges of colours (Green,Blue), or (Blue,Blue). From $\mathcal{C}_\infty$ we construct a new CDC $\mathcal{C}_\in$ of G_2 by 2.2.

3. Finally we construct G by adding the remaining edges. All the new edges now connect cycles in $\mathcal{C}_{\in}$ corresponding to cycles in H given by colours (Red,Blue) and we are done by Proposition 2.2.

∎

All the work done in this section is really based on the following observation. Let G_1 and G_2 be two graphs, $\mathcal{C}_1$ and $\mathcal{C}_2$ two CDC's of G_1 and G_2 respectively, and $C_1 \in \mathcal{C}_1$, $C_2 \in \mathcal{C}_2$ two cycles such that C_1 and C_2 have the same length. If we now construct a new graph by identifying the two cycles C_1 and C_2, and removing them from the CDC, we obtain a new graph and a CDC of this new graph as well. In the same way one can work with not just a pair of cycles but several pairs.

3. THE PROOF OF THE PUDDING, KOTZIG GRAPHS

In order to make use of theorem 2.1 to show that some graph H is a good frame we need to find a good CDC in H, preferably a CDC such that any pair on edges in H belongs to some cycle in the CDC. One large class of graphs which do have a CDC with this property is the class of kotzigian graphs, defined next.

Definition 3.1. Let G be a regular multigraph. A proper edge colouring of G is said to be a *Kotzig colouring* of G if the union of the edges in any two colour classes form a hamiltonian cycle in G.

A graph having a Kotzig colouring is said to be *kotzigian* or a *Kotzig graph*.

Kotzigian graphs were first introduced and studied by Anton Kotzig in [15]. Kotzig first called kotzigian graphs *Hamiltonsche Graphen* (hamiltonian graphs) and later on *strongly hamiltonian* graphs. In recent years kotzigian graphs have been known as graphs with perfect one-factorisations, however in both parts of this paper we will honour Kotzig and use the term kotzigian graphs.

In [15] Kotzig studied bipartite graphs and found that in order for such graphs to be kotzigian their order had to be congruent to 2 modulo 4. After this introduction of the concept Kotzig went on to show, in [16], that there exists two operations such that any cubic Kotzig graph can be constructed from an edge of multiplicity three by repeated use of these operations. Unfortunately these operations do not give Kotzig graphs but rather Kotzig coloured Kotzig graphs. Thus every Kotzig graph will be constructed with multiplicity equalling its number of Kotzig colourings, causing some headaches in an enumerative setting.

In [17] Kotzig gave his first written exposition in english concerning kotzigian graphs, both citing his earlier results and also introducing new theorems. He first gives a new set of three planarity preserving operations and show that any planar cubic Kotzig graph can be constructed using them. Next he shows that any cubic Kotzig graph with a specified red/blue/green Kotzig colouring can be viewed as an alternating euler tour of the red/green hamilton decomposition in the 4-regular graph obtained by contracting the blue edges, and vice versa. He also gives a few results concerning the number of Kotzig colourings of a graph. Finally he shows that if $m-1$ is an odd prime then K_m is kotzigian and goes on to state the well known, and still open, conjecture that K_{2n} is kotzigian for all $n > 1$.

Much later Kotzig and Labelle gave a further study of the structure of kotzigian graphs in [18].

For kotzigian graphs theorem 2.1 immediately gives us the following result.

THEOREM 3.1. *A kotzigian graph H is a 6-good frame.*

Proof. Since H is kotzigian it has a CDC consisting of the three 2-coloured hamiltonian cycles given by the Kotzig colouring and so the theorem follows from Proposition 2.1. ∎

Given one or several kotzigian graphs there are a number of ways in which we may obtain larger kotzigian graphs.

PROPOSITION 3.1. *The graph $\widehat{G}$ is constructed from a Kotzig graph G by replacing a vertex by a triangle as shown in Fig.2 is a Kotzig graph.*

This is is called a Δ-transformation of G.

Proof. Colour the edges not incident with the triangle in $\widehat{G}$ the same way as the corresponding edges in G, and the edges in the triangle according to figure 2. ∎

The previous proposition is in fact a special case of the following proposition.

PROPOSITION 3.2. *Let G_1 and G_2 be two Kotzig graphs. A graph H is called a* mating *of G_1 and G_2 if H can be constructed as follows. Let v_1 be a vertex in G_1 and v_2 a vertex in G_2. Form a new graph H as the vertex disjoint union of G_1 and G_2, delete v_1 from the copy of G_1 and v_2 from the copy of G_2. Add three new edges to H, each with one endpoint in both components of H, in such a way that H now is cubic.*

A mating of two Kotzig graphs is a Kotzig graph.

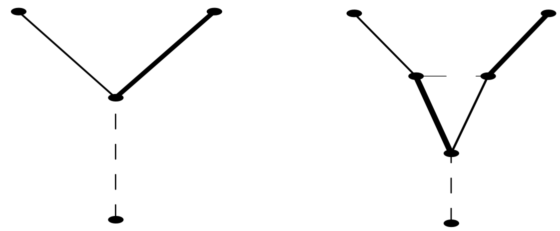

FIG. 2. Δ-transformation, with the local Kotzig colouring

Proof. Let the edges within the parts of H corresponding to G_1 and G_2 be coloured according to one of their respective Kotzig colourings and permute the colours in G_2 so that the colouring can be extended to the three new edges. This edge colouring will be a Kotzig colouring of H. ▮

A Halin graph is a graph constructed from a tree by taking a planar embedding of the tree and adding edges to form a cycle, whose vertices are the leaves of the tree, traversing the leaves in the order given by the planar embedding.

COROLLARY 3.1. *All cubic Halin graphs are kotzigian.*

Proof. K_4 is a Kotzig graph and any cubic Halin graph can be obtained from K_4 by repeated use of 3.1. ▮

COROLLARY 3.2. *Let G be a Kotzig graph and let $\hat{G}$ be the graph constructed from using the transformation in Fig 3, then $\hat{G}$ is a Kotzig graph.*

Proof. The graph $\hat{G}$ can be constructed as a mating of $K_{6,6}$ with G. ▮

PROPOSITION 3.3. *Let G be a Kotzig graph and let e_1 and e_2 be two edges with different colours in some Kotzig colouring of G. The graphs G_1 and G_2 obtained by using the transformations in Fig.4 on e_1 and e_2 are Kotzig graphs.*

Proof. Colour the edges not incident with the four-cycle in G_i the same way as the corresponding edges in G, and the edges incident with the four cycle according to figure 2. ▮

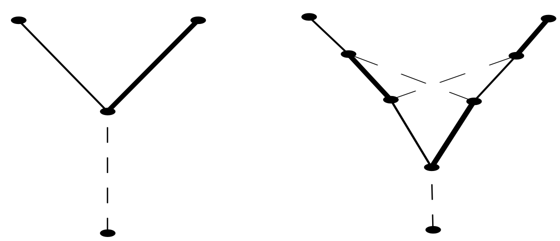

FIG. 3. Splitting a vertex.

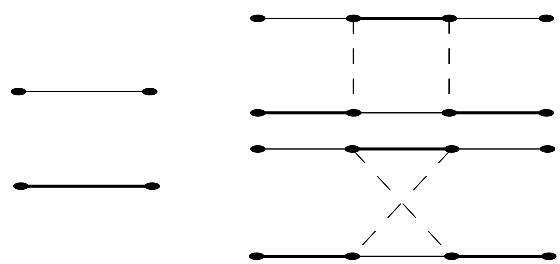

FIG. 4. Adding a four-cycle

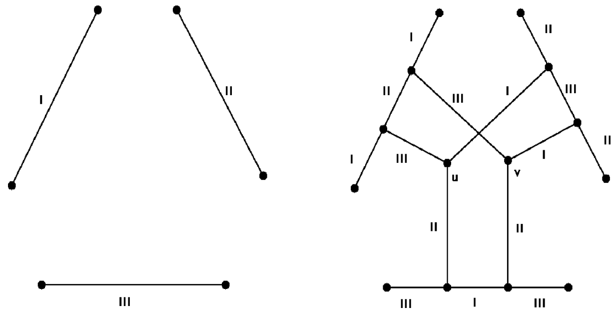

FIG. 5. Adding two independent vertices

The *Möbius ladder*, M_k,on k spokes is the graph obtained from C_{2k} by adding edges between vertices i and $i+k$ for all $i \leq k$.

COROLLARY 3.3. *M_k is kotzigian for odd k, $k \geq 3$.*

Proof. M_3 can easily be checked to be a Kotzig graph. For $k > 3$ the Möbius ladders can be constructed from M_3 by repeated use of 3.3. ▌

PROPOSITION 3.4. *Let G be a Kotzig graph, and let e_1, e_2, and e_3 be three edges in G with different colours in some Kotzig colouring of G.*

Let H be a graph constructed by first subdividing the edges e_i twice, see figure 5, then adding two new independent vertices u, v and adding edges so that both u and v are connected to a new vertex on each of the subdivided edges, and the graph H is cubic. Then H is a Kotzig graph.

Proof. By extending the colouring in figure 5 in the only way possible to get a proper three colouring of H we obtain a Kotzig colouring of H. In order to verify this just check to see that the original vertices are visited by each cycle in the same order as they were in the graph G. ▌

As a final example of Kotzig graphs we show in Fig 6 Kotzig colourings of the unique (3,6)-cage, the Heawood graph, and in Fig 7 a Kotzig colouring of the dodecahedron.

3.1. Enumeration of Kotzig graphs

One question that immediately becomes of interest when considering Kotzig graphs as frames is of course how common Kotzig graphs are among the cubic graphs on n vertices. In Table 1 we see the number of Kotzig graphs among the cubic graphs on n vertices for some small values of n. In fact using the operation from proposition 3.3 a lower bound on the number of labelled Kotzig graphs can be given. To give such bound we first observe

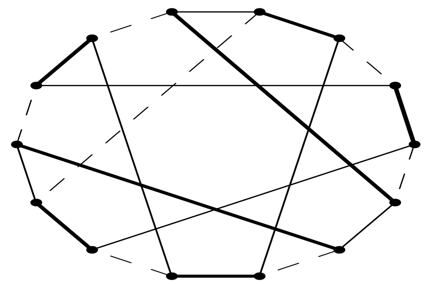

FIG. 6. A Kotzig colouring of the Heawood graph.

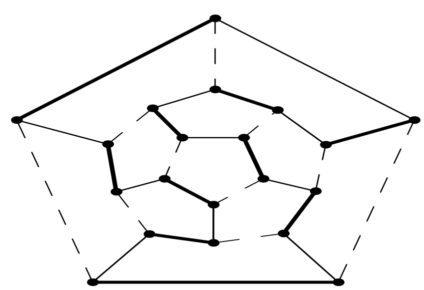

FIG. 7. A Kotzig colouring of the dodecahedron.

TABLE 1.

Counts of cubic graphs with additional properties. The percentages show Kotzig graphs among the hamiltonian graphs.

n	Kotzig	non-Kotzig	Hamiltonian	Connected
8	2(40%)	3	5	5
10	10(59%)	9	17	19
12	31(39%)	54	80	85
14	212(45%)	297	474	509
16	1614(42%)	2446	3841	4060
18	17708(46%)	23593	39454	41301

that in a Kotzig colouring of a cubic graph on n vertices there are $\frac{3}{4}n^2$ pairs of edges of different colours and secondly that using the operations of 3.3 increases the order of the graph by 4. Using this recursively we find that the number of kotzigian graphs has a superexponential lower bound, close to a sixth root of the number of cubic graphs. This is of course far from a sharp result.

Motivated by these numerical results and some heuristic probabilistic reasoning we make the following conjecture

Conjecture 3.1. Almost all cubic graphs are kotzigian.

This conjecture has also been formulated by Wormald in [28]. Newer results by Wormald actually implies that the proportion of kotzigian cubic graphs can not be less than $(\sqrt{n})^{-1}$, and bounds of second moment type would most likely turn this into a proof of the conjecture.

4. SWITCHABLE CDC'S

In order to be kotzigian it is necessary for a graph to be 3-connected, but we have another useful kind of CDC which is possible for 2-connected graphs as well, so called switchable CDC's.

DEFINITION 4.1. Let G be a 3-edge-colourable cubic graph. Assume that there is a 3-edge-colouring such that

1. Colours 1 and 2 together form a hamiltonian cycle.
2. Colours 1 and 3 together form a hamiltonian cycle.
3. Colours 2 and 3 form 2 disjoint two edge coloured cycles, C_1 and C_2.

Assume further that if colours 2 and 3 are exchanged on the edges of C_2 we get a new 3-edge colouring satisfying the three properties. Then each of these two CDC's are said to be a *switchable CDC*.

Using theorem 2.2 we get the following analogue of the result for Kotzig graphs.

THEOREM 4.1. *If A cubic graph H has a switchable CDC then H is* 6-*good frame.*

Proof. Let G be a cubic graph and M a matching such that $G \setminus M$ is a subdivision of H.

If there are no edges connecting C_1 and C_2 then G has a CDC by 2.1 and we are done. So we assume that there are k edges connecting C_1 and C_2.

If k is even we are done by 2.2. If there is an odd number of edges whose endpoints reside on edges of the same colour in H we can choose them to string the two hamiltonian cycles in H instead of connecting C_1 and C_2 and we are once more done by 2.2.

If k is odd and there is an odd number of edges whose endpoints reside on edges of different colour we can switch the colours in C_1 in order to get back to the preceding case and we are done once more.

To see that G has a 6-CDC we note that the cycles in a CDC of G coming from the two hamiltonian cycles in H give us at most four sets of disjoint cycles and the cycles with edges stringing or connecting C_1 and C_2 give us at most two further sets of disjoint cycles. ▮

Starting from pairs of Kotzig graphs we have several ways to construct graphs with switchable CDCs.

THEOREM 4.2. *Let G_1 and G_2 be two Kotzig graphs. Form a new graph H as follows. Let e_1 be an edge in G_1 and e_2 an edge in G_2. Delete e_1 and e_2 and add two new edges with one endpoint in G_1 and one endpoint in G_2 to obtain a new cubic graph H.*

The graph H has a switchable CDC.

Proof. Colour the copies of G_1 and G_2 red, green, and blue according to their respective Kotzig colouring and permute the colours so that the two new edges receive the same colour, say red.

Now (red,green) and (red,blue) will form two hamiltonian cycles in H. It is so because the removal of the edges from the hamiltonian cycles in G_1 and G_2 turn them into hamiltonian paths which are then linked to form a hamiltonian cycle of H.

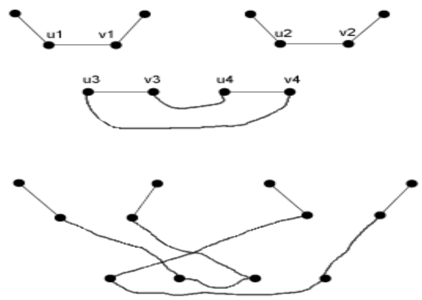

FIG. 8.

The (blue,green) edges will form two cycles, each a hamiltonian cycle in the copy of G_1 and G_2 respectively. If we interchange blue and green in G_1 we still have two hamiltonian cycles of H and a new hamiltonian cycle of G_1. ▮

THEOREM 4.3. *Let G_1 and G_2 be Kotzig graphs. Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be two edges with the same colour in some Kotzig colouring of G_1 such that the two hamiltonian cycles through e_1 and e_2 traverse e_1 and e_2 in the same direction, from u_i to v_i. Let $e_3 = (u_3, v_3)$ and $e_4 = (u_4, v_4)$ be two edges with the same colour in some Kotzig colouring of G_2.*

Now form a new graph H by removing the edges $e_1, \ldots, e_4$ from the disjoint union of G_1 and G_2, then add the edges $(u_1, v_3), (v_1, u_4), (u_2, u_3), (v_2, v_4)$. See Fig 8.

Then H has a switchable CDC.

Proof. The validity of the theorem can be seen in figure 8. Taking care to connect the new edges correctly with respect to the direction in which they traverse the removed edges we merge the subpaths of the two hamiltonian cycles in G_1 and G_2 into a hamiltonian cycle of H. Since the original hamiltonian cycles agreed in the direction in which they traversed the edges the construction will give two hamiltonian cycles in H. ▮

Note that both 4.2 and 4.3 are variations on the same underlying idea, to exchange an edge in a hamiltonian cycle in one graph for a path in another graph. Larger edge sets can be similarly handled, keeping track of the orientation of each relevant edge along the hamiltonian cycle. An example of this will be used in our analysis of the Petersen graph in a later section.

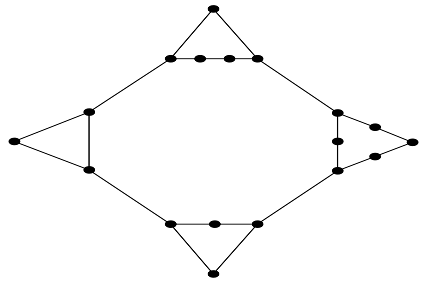

FIG. 9. An example of a 4-string of pearls.

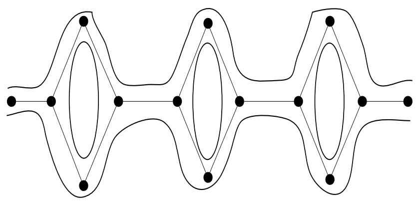

FIG. 10. A part of a CDC of a string of pearls.

5. ONE STEP FURTHER, FRAMES RELATED TO 2-FACTORS

We will now examine a few families of frames related to 2-factors and in a common way of mathematical writing we will begin not with the original idea but rather some generalisations of it. This is done in the hope of giving a more transparent description of the techniques used.

A *k-string of pearls* is a graph constructed as follows. Let $C_1, C_2, \ldots, C_k$ be k cycles and for each i add an edge from one vertex in C_i to one vertex in C_{i+1}, taking $k+1 = 1$, so that the resulting graph has maximum degree three, see Figure 9.

A CDC of a string of pearls has a quite simple structure, see Figure 10 for an illustration. It consist of all the cycles C_i together with two additional longer cycles. These longer cycles both use all the edges outside the C_i's, and their intersections with the C_i's partition the edges of each C_i into two paths. We will call the two long cycles C_{l1} and C_{l2}. We also note that at each C_i it does not matter which of the two long cycles uses which path in that C_i.

We can now formulate our first result along this line of thought.

THEOREM 5.1. *A string of pearls is a* 6*-good frame.*

Proof. Let G be a graph with a string of pearls H as a frame and fix a CDC $\mathcal{C}_1$ of H. In order to construct the sought CDC of G we shall partition the edges in $E(G) \setminus E(H)$ into five sets E_1, E_2, E_3, E_4, E_5.

1. Let E_1 contain all edges with at least one endpoint residing on an edge not in any of the C_i's.
2. Let E_2 contain all edges stringing any of the C_i's.
3. Form E_3 as follows. For each unordered pair (i, j) let $E_{(i,j)}$ be a maximum subset of even size of the edges connecting C_i and C_j. Put $E_3 = \bigcup E_{(i,j)}$.
4. The edges now remaining correspond to pairs of C_i's not connected by an even number of edges in H. To construct E_4 we now consider a graph G^* whose vertices are the C_i's and whose edges are the edges in $E(G) \setminus E(H)$ which are left so far. Now let E_4 be a subset of the edges in G^* such that $G^* \setminus E_4$ is a forest and E_4 forms a set of disjoint cycles in G^*. Call the forest F.
5. Let E_5 contain the edges in F.

Replace the CDC $\mathcal{C}_1$ with a new CDC $\mathcal{C}_2$ such that each edge in E_5 strings either C_{l1} or C_{l2}. That this is always possible follows from the fact that if we scan through each tree in F in a breadth first manner we can change the way C_{l1} and C_{l2} passes through the C_i's corresponding to the vertices in the tree. Since we are scanning through a tree our choices will never conflict.

We are now in the situation that an edge in $E_1 \cup E_5$ span either C_{l1} or C_{l2} and each C_i contains an even number of endpoints of edges in $E_{234} = E_2 \bigcup E_3 \bigcup E_4$.

We are now ready to describe the CDC $\mathcal{C}$ of G.

- i We first colour the edges in E_{234} red, and the segments between their endpoints blue and green. We add the resulting (red/blue) and (red/green) cycles to $\mathcal{C}$.
- ii Next for each of C_{l1} and C_{l2} we colour the edges in E_1 and E_5 red, the segments between their endpoints blue and green and add all the (red/blue), and (red/green) cycles to $\mathcal{C}$.

Each edge has now been covered twice and we are done. ∎

This class of frames can be extended significantly in the following way.

PROPOSITION 5.1. *Let $G_1, G_2, \ldots, G_k$ be kotzigian graphs with one edge deleted. Now add one edge from G_i to G_{i+1} to form a cubic graph H in the same way as in the construction of a string of pearls. The graph H is a* 6*-good frame.*

Proof. We first colour the G_i's in accordance with a Kotzig colouring of each graph, choosing the colours so that the deleted edges would have received the colour blue and let the other two colous be red and green. Next we colour the edges connecting the G_i's blue so that we get a three-colouring of H.

Now let G be a graph with a frame H as in the theorem. In order to find a CDC of G we proceed in the same way as in the proof of theorem 5.1, with the following modifications. First we replace the cycles C_{l1} and C_{l2} with the two cycles induced by the red/blue and the green/blue colourclases. Next, in the set E_1 we now include all edges with one endpoint on a blue edge. This can be done since all edges of this kind string the red/blue and green/blue cycles. Finally we let the red/green cycles play the role of the C_i's in the proof of theorem 5.1. ▮

In fact these are just the simplest members of a large class of recursively constructed good frames, generalising the class of kotzigian frames.

DEFINITION 5.1. Let $\mathcal{H}_0$ be the set of Kotzig graphs, each with a Kotzig colouring given. The colours are assumed to be red, green, and blue.

A graph H_2 belongs to $\mathcal{H}_{i+1}$ if it can be constructed from a graph H_1 in $\mathcal{H}_i$ by removing a blue edge from H_1 and a blue edge from a Kotzig graph H_3 and adding edges from the vertices of degree 2 in H_1 to the vertices of degree 2 in H_3 so that a new cubic graph is obtained. The two new edges are coloured blue.

A graph $G \in \mathcal{H}_i$, for some i is called an *iterated Kotzig graph.*

THEOREM 5.2. *An iterated Kotzig graph is a 6-good frame.*

Proof. The proof is identical to the proof of proposition 5.1 ▮

Something that should be noted here is that in the class of Kotzig graphs, H_0 in the theorem, we include graphs with multiple edges, most notably the graph with two vertices end three edges. This allows us to make a more complete connection to strings of pearls via substitutions of the kind used in the previous theorem.

The original motivation for considering these types of frames was the following construction.

DEFINITION 5.2. Let G be a cubic graph and C a 2-factor of G. Then G_C is the multigraph constructed by contracting each cycle in C to a vertex and removing all loops.

Theorem 5.1 can now be seen as a generalisation of the situation when G_C has a hamiltonian cycle, in the sense that a hamiltonian cycle in G_C implies a string of pearls as a frame but with the set E_1 empty. Let us formulate this as a separate observation, and also note that the graph G_C can be used when trying to decide whether a given graph G has other kinds of frames as well.

Proposition 5.2.

1. *If G_C has even degrees then G is 3-edge-colourable.*
2. *If G_C is hamiltonian then G has a CDC.*
3. *If G_C has a spanning cubic kotzigian subgraph then G has a kotzigian frame, and therefore has a CDC.*
4. *If G_C has a spanning cubic subgraph with a switchable CDC then G has a frame with a switchable CDC.*

Proof. Part one is trivial since the condition implies that G has a 2-factor with only even cycles. Part two is, as mentioned above, a corollary to theorem 5.1. Parts three and four are proven by replacing each vertex in the spanning subgraph by a triangle using the Δ-transformation. ▌

We close this section with a theorem of a more general nature, somewhat reminiscent of the theorems in section three of [31] giving a characterisation of graphs with 5, 6, and 7-CDCs.

Theorem 5.3. *Assume that G_C has a subgraph H with the following properties.*

1. *H contains all vertices in G_C of odd degree and has degree three at these vertices.*
2. *H has degree two at those vertices of even degree in G_C which it contains.*
3. *H has a k-CDC.*

Then G has a $(k+2)$-CDC.

Proof. If such a subgraph exists then the graph resulting from contracting one edge incident with every vertex of degree two in $G \setminus E(H)$ has an even 2-factor and is 3-edge-colourable. We colour the edges corresponding to C in this graph red and blue and the remaining edges green. Let $\mathcal{C}_1$ be the CDC given by this 3-edge-colouring and $\mathcal{C}'=\mathcal{C}_1 \setminus C$

Next we chose a k-CDC $\mathcal{C}_2$ of H and then replace the vertices in H by the corresponding cycles in G, adapting the CDC as shown in figure 11. Let $\mathcal{C}''=\mathcal{C}_2 \setminus C$ and finally take $\mathcal{C}=\mathcal{C}' \bigcup \mathcal{C}''$. Now $\mathcal{C}$ is our sought for CDC of G. ▌

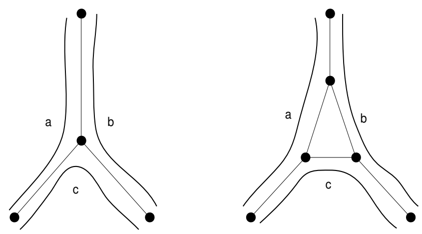

FIG. 11.

6. DISCONNECTED FRAMES

Of course frames do not need to be connected—older results in this vein are for instance the results on oddness, mentioned in the introduction. Saying that oddness k implies the existence of a CDC can be formulated as saying that a frame consisting of any number of even cycles and k triangles is a good frame. In our context we can also make use of different kinds of disconnected frames by observing that subgraphs with a bipartite 2-factor often do not need to be part of the connected component of a frame.

A generalisation of the case where G_C is hamiltonian requires not a spanning subgraph of G_C but instead a cycle containing the odd vertices in G_C.

THEOREM 6.1. *Assume that G can be partitioned into two induced subgraphs H_1 and H_2 such that H_1 has a bipartite 2-factor and H_2 has a string of pearls as frame. Then G has a 6-CDC.*

Proof. Let H be the string of pearls and fix a CDC $\mathcal{C}_1$ of H. Let $H_{1,1}, H_{1,2}, \ldots$ be the connected components of H_1, and let C be a bipartite 2-factor of H_1. Let C_{l1} and C_{l2} be the two long cycles in H, and let $C_1, C_2, \ldots$ be the short cycles in H.

In order to construct the sought CDC of G we will again partition the edges in $E(G) \setminus (E(C) \cup E(H))$ into eight sets E_k. The partition will be built in two steps.

1. Let E_1 contain all edges with at least one endpoint residing on an edge in $C_{l1} \cap C_{l2}$ and no endpoint in H_1.
2. Let E_2 contain all edges stringing any of the C_i's
3. Form E_3 as follows. For each unordered pair (i, j) let $E_{(i,j)}$ be a maximum subset of even size of the edges connecting C_i and C_j. Put $E_3 = \bigcup E_{(i,j)}$.
4. Form E_4 as follows. For each unordered pair (i, j) let $E_{(i,j)}$ be a maximum subset of even size of the edges connecting C_i and $H_{1,j}$. Put $E_3 = \bigcup E_{(i,j)}$.

5. The edges now remaining in H_2 correspond to pairs of C_i's not connected by an even number of edges in H. To construct E_5 we now consider a graph G^* whose vertices are the C_i's and the $H_{1,j}$'s, and whose edges are the edges left so far which have endpoints either on two C_i's or on one C_i and one $H_{1,j}$.

   Now let E_5 be a subset of the edges in G^* such that $G^* \setminus E_5$ is a forest and E_5 forms a set of cycles in G^*. Call the forest F.
6. Let E_6 contain the edges in F.

Now replace the CDC $\mathcal{C}_1$ with a new CDC $\mathcal{C}_2$ such that

(i) Each edge in E_6 strings either C_{l1} or C_{l2}. Let $E_{6,1}$ be the set of edges in E_6 stringing C_{l1} and let $E_{6,2}$ be the set of edges in E_6 stringing C_{l2}
(ii) If Λ_j is the set of edges with endpoints in both $H_{1,j}$ and H_2 such that $\{e \notin (E_4 \cup E_5\}$, then every edge in Λ_j connects $H_{1,j}$ to the same cycle C_{li}.

This change is always possible since by scanning through each tree in F in a breadth first manner we can change the way C_{l1} and C_{l2} passes through the C_i's corresponding to the vertices of the tree.

Next we partition the remaining edges in $E(G) \setminus (E(C) \cap E(H))$.

(i) If Λ_j is empty we add all edges in $H_{1,j} \setminus E(C)$ to E_7.
(ii) If the edges in Λ_j connects $H_{1,j}$ to C_{li} we partition the edges in $H_{1,j} \setminus E(C)$ into two subsets $E_{8,j,i}$ and $E_{8,j}$ such that each cycle in $C \cap H_{1,j}$ is incident with an even number of edges from $E_{8,j,i}$ and Λ_j. That this is possible follows from lemma 6.1 by considering the graph whose vertices are the cycles in $C \cap H_{1,j}$ and whose edges are the edges in $E(H_{1,j}) \setminus E(C)$.
(iii) Finally put $E_8 = \bigcup_j E_{8,j}$, $E_{8,1} = \bigcup_j E_{8,j,1}$, and $E_{8,2} = \bigcup_j E_{8,j,2}$.

Let

$$E_{23458} = E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_8,$$

$$E_{l1} = E_{6,1} \cup E_{8,1},$$

$$E_{l2} = E_{6,2} \cup E_{8,2}.$$

We are now ready to describe a CDC $\mathcal{C}$ of G.

(i) We first colour the edges in E_{23458} red, and the segments between their endpoints blue and green. We add the resulting (red/blue) and (red/green) cycles to $\mathcal{C}$.

(ii) Next we colour the edges in E_{l1} and the edges in E_1 which strings C_{l1} red, the segments between their endpoints blue and green, and add all the (red/blue), and (red/green) cycles to $\mathcal{C}$.
(iii) Next we colour the edges in E_{l2} and the edges in E_1 which strings C_{l2} red, the segments between their endpoints blue and green, and add all the (red/blue), and (red/green) cycles to $\mathcal{C}$.
(iv) Finally some of the cycles in $H_1 \cap C$ may have been covered only once, in which case we add these cycles to $\mathcal{C}$.

Each edge has now been covered twice and we are done. ∎

In order to state the lemma used in the previous proof we need the following definition.

DEFINITION 6.1. Let G be a graph and T a subset of its vertices. A set $F \subseteq E(G)$ is called $T - join$ if the number of edges of F incident with a vertex v in G is odd if $v \in T$ and even otherwise.

LEMMA 6.1. *A connected graph G possesses a $T - join$ if and only if T contains an even number of vertices.*

For a proof and more facts about T-joins see [1].

Continuing in this way we can make use of both partial kotzigian frames and partial frames with switchable CDCs as well.

THEOREM 6.2. *If G can be partitioned into two induced subgraphs H_1 and H_2 such that H_1 has a bipartite 2-factor and H_2 has an iterated Kotzig graph as a frame, then G has a 6-CDC.*

Proof. Let H be the iterated Kotzig graph, coloured red, green, and blue, and fix a CDC $\mathcal{C}_1$ of H. Let $H_{1,1}, H_{1,2}, \ldots$ be the connected components of H_1, and let C be a bipartite 2-factor of H_1. Let C_{l1} and C_{l2} be the two long cycles in H, given by the red/blue and green/blue edges respectively, and let $C_1, C_2, \ldots$ be the short, red/green, cycles in H.

In order to construct the sought CDC of G we will once more partition the edges in $E(G) \setminus (E(C) \cup E(H))$ into nine sets E_k. The partition will be built in two steps.

1. If there is an edge with one endpoint in $H_{1,j}$ and one endpoint on a blue edge in H let $E_{0,j}$ contain all edges with one endpoint in $H_{1,j}$ and one endpoint in H_2.
2. Let E_1 contain all edges with at least one endpoint residing on an edge in $C_{l1} \cap C_{l2}$ and no endpoint in H_1.
3. Let E_2 contain all edges stringing any of the C_i's

4. Form E_3 as follows. For each unordered pair (i, j) let $E_{(i,j)}$ be a maximum subset of even size of the edges connecting C_i and C_j. Put $E_3 = \bigcup E_{(i,j)}$.
5. Form E_4 as follows. For each unordered pair (i, j) let $E_{(i,j)}$ be a maximum subset of even size of the edges connecting C_i and $H_{1,j}$. Put $E_4 = \bigcup E_{(i,j)}$.
6. The edges now remaining in H_2 correspond to pairs of C_i not connected by an even number of edges in H. To construct E_5 we now consider a graph G^* whose vertices are the C_i's and the $H_{1,j}$'s, and whose edges are the edges left so far which have endpoints either on two C_i's or on one C_i and one $H_{1,j}$.

   Now let E_5 be a subset of the edges in G^* such that $G^* \setminus E_5$ is a forest and E_5 forms a set of cycles in G^*. Call the forest F.
7. Let E_6 contain the edges in F.

Now replace the CDC $\mathcal{C}_1$ with a new CDC $\mathcal{C}_2$ such that

(i) Each edge in E_6 strings either C_{l1} or C_{l2}. Let $E_{6,1}$ be the set of edges in E_6 stringing C_{l1} and let $E_{6,2}$ be the set of edges in E_6 stringing C_{l2}

(ii) If Λ_j is the set of edges with endpoints in both $H_{1,j}$ and H_2 such that $\{e \notin (E_4 \cup E_5\}$, then every edge in Λ_j connects $H_{1,j}$ to the same cycle C_{li}.

This change is always possible since by scanning through each tree in F in a breadth first manner we can change the way C_{l1} and C_{l2} passes through the C_i's corresponding to the vertices of the tree.

Next we partition the remaining edges in $E(G) \setminus (E(C) \cap E(H))$.

(i) If Λ_j is empty we add all edges in $H_{1,j} \setminus E(C)$ to E_7.

(ii) If the edges in Λ_j connects $H_{1,j}$ to C_{li} we partition the edges in $H_{1,j} \setminus E(C)$ into two subsets $E_{8,j,i}$ and $E_{8,j}$ such that each cycle in $C \cap H_{1,j}$ is incident with an even number of edges from $E_{8,j,i}$ and Λ_j. That this is possible follows from lemma 6.1 by considering the graph whose vertices are the cycles in $C \cap H_{1,j}$ and whose edges are the edges in $E(H_{1,j}) \setminus E(C)$.

(iii) Let $E_8 = \bigcup_j E_{8,j}$, $E_{8,1} = \bigcup_j E_{8,j,1}$, and $E_{8,2} = \bigcup_j E_{8,j,2}$.

Finally we partition the edges the $E_{0,j}$'s into two subsets,

1. If $H_{1,j}$ is connected to only one of the C_{li}:s we add all edges in $E_{0,j}$ to E_0^i.
2. If If $H_{1,j}$ is connected to both C_{l1} and C_{l2} we half of the edges in $E_{0,j}$ to E_0^1 and half to E_0^2. This division can be done arbitrarily.

Let

$$E_{23458} = E_2 \cup E_3 \cup E_4 \cup E_5 \cup E_8,$$

$$E_{l1} = E_{6,1} \cup E_{8,1} \cup E_0^1,$$

$$E_{l2} = E_{6,2} \cup E_{8,2} \cup E_0^2.$$

We are now ready to describe a CDC $\mathcal{C}$ of G.

(i) We first colour the edges in E_{23458} red, and the segments between their endpoints blue and green. We add the resulting (red/blue) and (red/green) cycles to $\mathcal{C}$.

(ii) Next we colour the edges in E_{l1} and the edges in E_1 which strings C_{l1} red, the segments between their endpoints blue and green, and add all the (red/blue), and (red/green) cycles to $\mathcal{C}$.

(iii) Next we colour the edges in E_{l2} and the edges in E_1 which strings C_{l2} red, the segments between their endpoints blue and green, and add all the (red/blue), and (red/green) cycles to $\mathcal{C}$.

(iv) Finally some of the cycles in $H_1 \cap C$ may have been covered only once, in which case we add these cycles to $\mathcal{C}$.

Each edge has now been covered twice and we are done. ∎

THEOREM 6.3. *If G can be partitioned into two induced subgraphs H_1 and H_2 such that H_1 has a bipartite 2-factor and H_2 has a frame with a switchable CDC, then G has a 6-CDC.*

Proof. Let H be a frame of H_2 which has a switchable CDC, let the two long cycles in this CDC be C_{l1} and C_{l2}, and let C_1 and C_2 be the two short cycles. Let C be the bipartite 2-factor of H_1.

We first note that there is an even number of edges connecting each component of H_1 to H_2.

For each component H_{1i} of H_1 there are three possibilities.

1. There is at least one edge e with one endpoint in H_{1i} and one endpoint on an edge in $C_{l1} \cup C_{l2}$.

   By choosing which of C_{l1} and C_{l2} the edge e connects H_{1i} to, we can make sure that H_{1i} is connected by an even number of edges to both C_{l1} and C_{l2}.
2. All edges with endpoints in both H_{1i} and H_2 have their endpoints on C_1 and C_2 and there is an even number of edges connecting each of C_1 and C_2 to H_{1i}.
3. All edges with endpoints in both H_{1i} and H_2 have their endpoints on C_1 and C_2 and there is an odd number of edges connecting each of C_1 and C_2 to H_{1i}.

Next we choose each edge in H_2 with at least one endpoint on an edge in $C_{l1} \cup C_{l2}$ to string one of C_{l1} and C_{l2}.

If there is an odd number of edges connecting C_1 and C_2 and an even number of components of type 3 above we can switch the CDC of H in order to make sure that for each of C_1 and C_2 there is an even number of edges with endpoints on that cycle.

If there is an even number of edges connecting C_1 and C_2 and an odd number of components of type 3 above we can likewise switch the CDC of H in order to make sure that for each of C_1 and C_2 there is an even number of edges with endpoints on that cycle.

On each of C_1 and C_2 we can now colour the paths between the endpoints of the edges that connect the given C_i either to H_1 or the other C_i alternatingly blue and green. We then colour the edges in $C \cup H_{1i}$ alternatingly blue and green for each H_{1i} of type 1 or 2 and the remaining edges in these H_{1i}'s red. Finally we colour the edges connecting C_1 and C_2 red. Now the 3-CDC given by the red/green, red/blue, and green/blue paths contains the subdivisions of both C_1 and C_2. Let $\mathcal{C}_1$ be this CDC but with C_1 and C_2 removed.

Next for each component H_{1i} connected to both C_{l1} and C_{l2} we partition the edges in $H_{1i} \setminus C$ into two subsets E_1 and E_2 such that each cycle in $C \cup H_{1i}$ is incident with an even number of edges either either in E_1 or connecting H_{1i} to C_{l1}.

For each of C_{l1} and C_{l2} we now perform a colouring. First colour each edge in E_j and each edge connecting H_{1i} to C_{lj} red. Colour the paths between the endpoints of edges either in E_1 or connecting H_{1i} to C_{lj} alternatingly blue and green. If we now take the blue/green, red/green and red/blue cycles from each of these colourings and remove the copies of each cycle present in both colourings we get a collection of cycles covering each edge in C_1 and C_2 once and every other edges twice. Call this collection of cycles $\mathcal{C}_2$.

We are now done since $\mathcal{C}_1 \cup \mathcal{C}_2$ forms our sought for 6-CDC. ∎

As we have seen it is of great interest to find subgraphs of G_C which contain all vertices of odd degree in G_C , in particular cycles. This immediately leads us into the field of results stemming from Dirac's theorem,

THEOREM 6.4 ([5]). *If G is k-connected, $k \geq 2$, then any two edges and any $k-2$ vertices in G lie on a common cycle.*

Combining this result with the theorems of this section we get,

COROLLARY 6.1. *If G_C has k vertices of odd degree and is k-connected, then G has a CDC.*

Corollary 6.2. *If G_C has k vertices of odd degree, at least two edges have both endpoints of odd degree, and G is $(k-2)$-connected, then G has a CDC.*

Using the following theorem by Häggkvist and Thomassen,

Theorem 6.5 ([9]). *If G is k-connected then there is a cycle through any $(k-1)$-matching in G*

we find

Corollary 6.3. *If G_C is k-connected and there is a matching of size at most $k-1$ containing all odd vertices in G_C then G has a CDC.*

As we can see just about every result guaranteeing a cycle through some set of vertices, edges or subgraph can be brought into play giving further restrictions of graphs without CDCs. For a nice survey of useful results on this topic see [2]. All said and done a possible counterexample to the cycle double cover conjecture must be very strange creature indeed.

7. CONJECTURES.

Considering the result in section 3 showing that the Möbius ladders M_k are kotzigian for odd k and not for even k the following would seem likely:

Conjecture 7.1. There is a j such that the Möbius ladders M_k are j-good frames for all even k.

On a grander scale we offer the following conjecture, which together with the results in this paper implies the cycle double cover conjecture.

Conjecture 7.2. All 3-connected cubic graphs have a Kotzig graph as a frame.

That 3-connected can not be replaced by 2-connected can be seen from the example in figure 12.

A simpler problem is probably the following,

Conjecture 7.3. If G is a cubic 2-connected graph on n vertices, then there exists a set of at most $\log_2 n$ edge pairs in G such that if each edge in these pairs is subdivided once and a new edge is added between the new vertices in each pair, then the graph so obtained is a Kotzig graph.

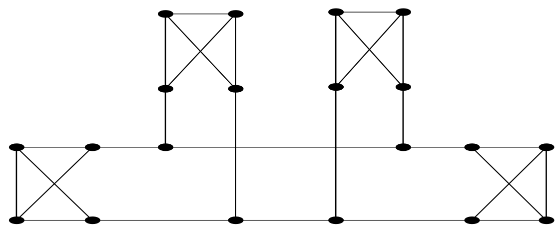

FIG. 12. A two-connected graph without a Kotzig frame.

Coming back to theorem 2.1 we also offer the following conjecture

Conjecture 7.4. If H is a cubic edge 3-colourable graph then H is a 6-good frame.

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