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Stability versus Hyperbolicity in Dynamical and Iterated Function Systems

Abstract

In this paper we investigate a certain notion of stability, for one function or for iterated function systems, and discuss why this notion can be a good extension and complement to the notion of hyperbolicity. This last notion is very well-known in the literature and plays an important role in the investigation of the dynamical behaviour of a system. The main result is that although some classical sets of functions like the stable Lipschitz functions are conjugate to hyperbolic functions there exist continuous stable functions which are not conjugate to hyperbolic functions. A sufficient condition for not being conjugate to a hyperbolic function is given.

1 Introduction

1.1

Let K be a nonempty compact subset of the plane $\mathbb{R}^2$ with the ordinary Euclidian metric d and $\mathcal{F}$ be a family of functions $f : K \to K$. We refer to $\{\mathcal{F}, K\}$ as an *iterated function system* (IFS). When $\mathcal{F}$ consists of one function $f : K \to K$ we also refer to $\{\mathcal{F}, K\}$ or $\{f, K\}$ as a (discrete) *dynamical system*. We remind of the following definition (see [Bar] or [De]).

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Definition 1.1. The system $\{\mathcal{F}, K\}$ is *hyperbolic* if there exists a $\lambda < 1$ such that

$$d(f(x), f(y)) \leq \lambda d(x, y),$$

for all $f \in \mathcal{F}$ and all points $x, y \in K$. If a hyperbolic system consists of a single function $f : K \to K$, the function f is called hyperbolic.

In order to introduce the notion of stability we need some more definitions. Let $\mathcal{F}^\infty$ denote the family of infinite sequences $\{f_n\}_{n=1}^\infty$ of functions in $\mathcal{F}$. If $F = \{f_n\}_{n=1}^\infty$ is an element in $\mathcal{F}^\infty$, then for each positive integer we introduce the composite function

$$F_n = f_1 \circ f_2 \circ ... \circ f_n = f_1(f_2(...(f_n)...)).$$

Observe that the sequence $F_n(K), n \geq 1$, where $F_n(K)$ is the image of K under F_n, is a nested, decreasing sequence, i.e. $F_{n+1}(K) \subset F_n(K)$ for $n \geq 1$.

Definition 1.2. The IFS $\{\mathcal{F}, K\}$ is *stable* if, for every element $F \in \mathcal{F}^\infty$, we have $\text{diam}(F_n(K)) \to 0$ as $n \to \infty$. (Here diam stands for the diameter.) If a stable system $\{\mathcal{F}, K\}$ consists of a single function f, then f is called stable on K.

In other words the system $\{\mathcal{F}, K\}$ is stable if and only if the nested sequence of sets $F_n(K), n \geq 1$, has a one-point set as its limit. Obviously, a hyperbolic system is stable but, as we shall see below, there are plenty of stable systems which are not hyperbolic. In fact we will show that functions which either are monotone or belong to the Lipschitz class are conjugate to hyperbolic functions (Th. 2.4 and 2.8) and both give a condition sufficient for not being conjugate and an example of a continuous function satisfying this condition (Th. 2.1).

The notion of stability was introduced in [Ka-Wa] in connection with a study of continued fractions. The definition there is somewhat stronger and following the terminology of that paper, we refer to that stronger form of stability as uniform stability:

Definition 1.3. ([Ka-Wa], Definition 1 and Proposition 1) An IFS $\{\mathcal{F}, K\}$ is *uniformly stable* if there exists a sequence of positive numbers $\{\rho_n\}$ tending to zero such that for any element $F \in \mathcal{F}^\infty$ and any positive integer n we have $\text{diam}(F_n(K)) \leq \rho_n$.

Evidently, a uniformly stable system is stable. It also turns out (see Proposition 2.9 in Section 2.3) that a stable system $\{\mathcal{F}, K\}$ is uniformly stable if $\mathcal{F}$ consists of a finite number of functions but not necessarily if the family $\mathcal{F}$ is infinite (see Example 2.11 in Section 2.3).

The above notions of hyperbolicity and stability can be defined for more general phase spaces K than compact subsets of the Euclidian plane but this is not our goal in the present paper. Here our aim is to compare hyperbolicity and stability from different points of view and to show that in many respects the notion of stability has advantages.

We first observe that stable dynamical systems in many respects have the same qualitative dynamical behaviour as hyperbolic systems. More exactly, for hyperbolic systems $\{\mathcal{F}, K\}$, where $\mathcal{F}$ is a finite family, a lot of properties are described for instance in [Bar] and [Hu], to which we refer for details. For $F = \{f_n\}_1^\infty \in \mathcal{F}^\infty$ there exists an attractor to which the orbit $\tilde{F}_n = f_n \circ f_{n-1} \circ ... \circ f_1, n \geq 1$ is attracted as n tends to infinity, and the reversed iterates $F_n = f_1 \circ f_2 \circ ... \circ f_n, n \geq 1$, converge to a point in the attractor. Furthermore, if $\{\mathcal{F}, K\}$ is an IFS with probabilities there exists an invariant probability measure supported by the attractor giving the relative asymptotic distribution of the points of an orbit. In [Ka-Wa] it is proved that the same results hold for uniformly stable systems.

In [Ka-Wa] the notion of stable dynamical systems was introduced in the study of convergence of certain continued fractions which were thought of as iterations of Möbius transformations. Convergence was proved by showing the stability property for these Möbius transformations. Other references relevant for the study of stability are [Ga], [St], [Öb], and [Am].

1.2

We now compare hyperbolic and stable systems in some different respects. We first discuss the important notion of conjugation.

Definition 1.4. An IFS $\{\mathcal{F}, K\}$ is *conjugate* to an IFS $\{\mathcal{F}', K'\}$ if there exists a homeomorphism $h : K \to K'$ such that

$$\mathcal{F}' = h \circ \mathcal{F} \circ h^{-1} = \{h \circ f \circ h^{-1} : f \in \mathcal{F}\}.$$

If $\{\mathcal{F}, K\}$ consists of a single function f we say that f is conjugate to $h \circ f \circ h^{-1}$.

Conjugate systems qualitatively have the same dynamical behaviour. Observe that if $\{\mathcal{F}, K\}$ is conjugate to a hyperbolic system then all $f \in \mathcal{F}$ are continuous functions, since a hyperbolic system consists of continuous functions.

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Stability, but not hyperbolicity, is invariant under conjugation. In fact, it can be checked (see Proposition 2.12, Section 2.3) that the stability property is invariant under conjugation. We now give an example showing that the hyperbolicity property of an IFS is not always preserved under conjugation.

Example 1.5. Consider the following standard example of an iterated function system

$$\mathcal{F} = \{f_1, f_2\},\ f_1(x) = \frac{1}{3}x,\ f_2(x) = \frac{1}{3}x + \frac{2}{3},$$

given on the unit interval $K = [0, 1] \subset \mathbb{R}$. This is a hyperbolic system. Now let us bend the interval [0,1] into an arc of degree 300° of a circle, and consider the IFS on that arc induced by that on the interval [0,1]. Evidently, the new IFS is conjugate to the old system, but the new one is no longer hyperbolic (the endpoints of the arc move away from each other under each of the mappings in the conjugate system).

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Stability, but not hyperbolicity, is a local property, if the phase space K is connected. To be more precise, a function can be locally hyperbolic (i.e. hyperbolic in a neighbourhood of each point) without being hyperbolic globally. The functions in the IFS in Example 1.5 defined on the arc, are examples of such functions. On the other hand, stability is a local property (see Proposition 2.9 in Section 2.3) where local stability is defined as follows.

Definition 1.6. An IFS $\{\mathcal{F}, K\}$ is *locally stable* if for any element $F \in \mathcal{F}^\infty$ and any point $x \in K$ there exists an open neighbourhood U of x such that $\mathrm{diam}(F_n(U)) \to 0$ as $n \to \infty$.

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The family of stable systems is richer than that of hyperbolic systems in the sense that there are stable systems which are not conjugate to any hyperbolic systems. Let us be more precise. In Example 1.5 we constructed an IFS on an arc of a circle, which is a stable but not a hyperbolic system. That system is conjugate to a hyperbolic system and consequently a natural question arises: Is every stable system a "damaged" hyperbolic system in the sense of conjugation?

A hyperbolic system consists of continuous functions and because of that a system conjugate to a hyperbolic system must consist of continuous functions only. Hence, a stable system containing at least one discontinuous function cannot be conjugate to a hyperbolic system. In the following example we construct a discontinuous stable function and this, consequently, shows that there are discontinuous stable functions which are not conjugate to hyperbolic functions.

Example 1.7. We shall construct a discontinuous, stable function f on the phase space $K = [0,1]$. We define f by putting $f(0) = 0$ and

$$f(x) = 2^{-n} \text{ for } 2^{-n} < x \le 2^{-n+1},\ n = 1, 2, \ldots$$

Then the function $f : [0,1] \to [0,1]$ is discontinuous and it is stable since if f^n is f iterated n times, then $f^n([0,1]) \subset [0, 2^{-n}]$, i.e. $\mathrm{diam}(f^n([0,1])) \to 0$. Obviously, by varying this construction we easily get a lot of examples of stable systems consisting of discontinuous functions.

The question arises: are there stable systems, consisting of continuous functions only, which are *not* conjugate to hyperbolic systems? In Theorem 2.1 in Section 2.1, the main result of this paper, we state that there are continuous stable functions which are not conjugate to any hyperbolic functions, i.e. stability is a fundamentally new property as compared to hyperbolicity even if we restrict ourselves to continuous functions. We also give some sufficient conditions for a continuous stable function to be conjugate to a hyperbolic function (see Theorems 2.4 and 2.8 in Section 2.2).

iv

The notion of stability can be generalised to topological spaces. In fact, the alternative form of stability indicated immediately after Definition 1.2 (that the nested sequence $F_n(K), n \ge 1$, has a one-point set as its limit) does not require a metric structure. In the context of general topological spaces it might however be more natural to demand that the sequence of sets has *at most* a one-point set as its limit. This change is prompted by the fact that in spaces which are not sequentially compact such a limit set might not exist.

2 Results

In Section 2.1 we discuss the existence of continuous stable functions which *are not* conjugate to hyperbolic functions and in Section 2.2 we treat some classes of continuous stable functions which *are* conjugate to hyperbolic functions. In Sections 2.1 and 2.2 the phase space is a compact interval of the real line and as a normalisation we choose the interval to be [0,1]. In Section 2.3, finally, we state three general properties of stable systems.

2.1

Our main result is the following theorem.

Theorem 2.1. *There exists a continuous function $f : [0,1] \to [0,1]$ which is stable, but which is not conjugate to any hyperbolic function.*

We shall see that Theorem 2.1 is a special case of Theorem 2.2 below. In the statement of Theorem 2.2 we need the following notation. Given a function f on [0,1], let $s_f(y)$ be the number of points $x \in [0,1]$ such that $f(x) = y$.

Theorem 2.2. *(a) Let $f : [0,1] \to [0,1]$ be a continuous, non-constant function such that $s_f(y) = \infty$ for all but countably many y in $f([0,1])$. Then f can not be conjugate to a function of bounded variation.*

(b) There exists a stable function $f : [0,1] \to [0,1]$ satisfying the conditions in part (a) of this theorem.

It is easy to see that Theorem 2.1 follows from Theorem 2.2. In fact, Theorem 2.2 implies that there exists a continuous stable function f which is not conjugate to any function of bounded variation and hence not to any hyperbolic function since a hyperbolic function on [0,1] is of bounded variation.

Remark. The function f constructed in the proof of (b) in Theorem 2.2 is an example of a so called Gillis function which is a function such that every level set of the function is a perfect set. The first example of such a function was given in [Gi]. The function f constructed can be shown to be nowhere differentiable but according to a theorem by Bari [Ba] we can find a homeomorphism h such that $g = h \circ f$ is differentiable almost everywhere. The new function g is still not conjugate to a hyperbolic function. For a further discussion see [Br-J].

Remark. We observe that every homeomorphism $h : [0,1] \to [0,1]$ gives a change of metric from the original metric d to a metric d' defined by $d'(x,y) = d(h(x),h(y))$. However, not every change of metric is given by a homeomorphism (see Example 2.3 below). Thus we can ask whether it is always possible to choose a new metric for [0,1] such that a given stable function f is hyperbolic in this new metric. If the new metric induces a measure of finite mass to the interval [0,1] a mapping which is hyperbolic in this new metric will also be of bounded variation and, consequently, the function in Theorem 2.2.(b) will still work as a counterexample. Hence, the class of continuous stable mappings is strictly larger than the class of hyperbolic mappings under these transformations as well.

Example 2.3. Let h be a homeomorphism from [0,1] to itself and let $0 < a < b < c < 1$. In the usual metric $d(x,y) = |x-y|$ we have $d(a,c) = d(a,b)+d(b,c)$ and this property holds for the metric $d'(x,y) = d(h(x),h(y))$ as well. Now we introduce the metric ρ by $\rho(x,y) = |x-y|$, if $x, y \in [0, \frac{1}{2}]$ or $x, y \in [\frac{1}{2}, 1]$, and by $\rho(x,y) = \left((\frac{1}{2} - x)^2 + (y - \frac{1}{2})^2\right)^{\frac{1}{2}}$ if $x \in [0, \frac{1}{2}]$ and $y \in [\frac{1}{2}, 1]$. In this metric we have that

$$\rho\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{\sqrt{2}}{4} < \rho\left(\frac{1}{4}, \frac{1}{2}\right) + \rho\left(\frac{1}{2}, \frac{3}{4}\right) = \frac{1}{2}$$

and, consequently, this metric can not be induced by a homeomorphism on [0,1].

Remark. By using methods similar to those used in [Br], pages 144-146 it can be proved that the set of continuous functions which are not conjugate to hyperbolic functions forms a residual subset of the set of all stable mappings.

2.2

Below we show that for some important classes of continuous functions stable functions in these classes are conjugate to hyperbolic functions. We say that a function f, defined on an interval $[a, b]$, belongs to $\mathrm{Lip}[a, b]$ if, for some positive constant M, the Lipschitz constant,

$$|f(x) - f(y)| \leq M|x - y|, \text{ for all } x, y \in [a, b].$$

Theorem 2.4. *Let $f : [0, 1] \to [0, 1]$ be a stable function on $[0, 1]$. If $f \in \mathrm{Lip}[0, 1]$, then f is conjugate to a hyperbolic function on $[0, 1]$.*

Theorem 2.4 is true in a slightly more general form.

Theorem 2.5. *Let $f : [0, 1] \to [0, 1]$ be a continuous stable function on $[0, 1]$ with fixed point x_0 and let $B_r(x_0) = \{x : |x - x_0| < r\}$. If $f \in \mathrm{Lip}([0, 1] \setminus B_r(x_0))$ for every r, $0 < r < 1$, then f is conjugate to a hyperbolic function on $[0, 1]$.*

Example 2.6. All the functions in Theorem 2.4 are of bounded variation but there are functions satisfying the conditions in Theorem 2.5 which are of unbounded variation, i.e. there are stable functions of unbounded variation which are conjugate to hyperbolic functions. We now construct such a function $f : [0, 1] \to [0, 1]$. On the interval $[\frac{1}{n+1}, \frac{1}{n}]$ where $n \geq 1$, let the graph of f be an isosceles triangle with base $[\frac{1}{n+1}, \frac{1}{n}]$ whose equal sides have length $\frac{1}{3n}$, and let $f(0) = 0$. It is easy to check that f is of unbounded variation and satisfies the conditions in Theorem 2.5.

Example 2.7. In Theorems 2.4 and 2.5 we use conditions on the rate of growth of the functions (Lipschitz conditions). The following construction shows that a fast rate of growth itself is not an obstacle to being conjugate to a hyperbolic function. Let, for $x \in [0, 1]$,

$$f(x) = \inf\left\{\frac{x}{2}, \left|x - 2^{-1}\right|, \left|x - 2^{-2}\right|^{\frac{1}{2}}, ..., \left|x - 2^{-n}\right|^{\frac{1}{n}}, ...\right\}.$$

This function f obviously does not belong to any Hölder class on [0,1]. It can be checked that $g = h \circ f \circ h^{-1}$, where $h(x) = \exp(-\frac{1}{x})$, $x \neq 0$, $h(0) = 0$, fulfils the conditions on f in Theorem 2.5. Because of that g and hence also f is conjugate to a hyperbolic function on [0,1]. By a simple modification functions f can be constructed which have arbitrary high growth rate and are conjugate to hyperbolic functions.

Theorem 2.8. *Let f be a continuous stable function. If f is one-to-one, then f is conjugate to a hyperbolic function on* $[0, 1]$.

Remark. The techniques in Section 3 used to prove Theorems 2.4, 2.5 and 2.8 depend in an essential way on the 1-dimensional structure of the real line. It is reasonable to guess that in higher dimensions the set of mappings conjugate to hyperbolic mappings will be relatively smaller than in one dimension.

2.3

We now finally formulate the three propositions and the example referred to in Section 1.

Proposition 2.9. *If K is a connected compact subset of the Euclidian plane, then any locally stable IFS $\{\mathcal{F}, K\}$ is a stable system.*

Proposition 2.10. *If K is a compact subset of the Euclidian plane and $\{\mathcal{F}, K\}$ is a stable IFS and $\mathcal{F}$ consists of a finite number of functions, then $\{\mathcal{F}, K\}$ is uniformly stable.*

The following example shows that for an IFS where the family $\mathcal{F}$ is infinite the assertion in Proposition 2.10 is not necessarily true, not even if $\mathcal{F}$ consists of continuous functions only.

Example 2.11. Let $\Delta_j = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1,\ x = jy\}$, $j = 1, 2, ...$, be a family of pairwise different diameters of the closed unit disc in the Euclidian plane centred at the origin. Let K be the closure of the union of all Δ_j. For each $j = 1, 2, ...$ we define a function f_j in the following way: $f_j(x, y) = 2^{-\frac{1}{j}}(x, y)$ if $(x, y) \in \Delta_j$, and $f_j(x, y) = (0, 0)$ otherwise. Evidently f_j is a continuous function on K. Let $\mathcal{F}$ consist of the functions $f_j, j = 1, 2, ...$ Then the IFS $\{\mathcal{F}, K\}$ is stable since $(f_j \circ f_i)(K) = \{(0, 0)\}$ if $j \neq i$, and, if f_j^n denotes f_j iterated n times, $\mathrm{diam}(f_j^n(K)) = 2 \cdot 2^{-\frac{n}{j}} \to 0$ as $n \to \infty$. However, the IFS is not uniformly stable since $\mathrm{diam} f_j^j(K) = 1$.

Proposition 2.12. *An IFS conjugate to a stable (respectively uniformly stable) system is stable (respectively uniformly stable).*

3 Proofs

Proof of Theorem 2.2.a . In the proof we shall use a theorem by Banach (see [Sa] page 280) stating that if f is a continuous real-valued function on $[0,1]$ and $V_0^1(f)$ is the total variation of f on $[0,1]$ then

$$V_0^1(f) = \int_{-\infty}^{\infty} s_f(y)dy.$$

Let f be a continuous, non-constant function from $[0,1]$ into itself and h a homeomorphism on the same interval. Since h is 1-1 we have that

$$s_{f\circ h}(y) = s_f(y)$$

$$s_{h^{-1}\circ f}(y) = s_f(h(y)).$$

According to the theorem by Banach referred to above we thus have that

$$V_0^1(h^{-1}\circ f\circ h) = \int_{-\infty}^{\infty} s_{h^{-1}\circ f\circ h}(y)dy =$$
$$= \int_{-\infty}^{\infty} s_{h^{-1}\circ f}(y)dy = \int_{-\infty}^{\infty} s_f(h(y))dy.$$

Now since $s_f(y) = \infty$ for all but countably many y the last integrand is infinite a.e. which means that the integral itself is infinite. We thus have that $V_0^1(h^{-1}\circ f\circ h) = \infty$ for all homeomorphisms h. Hence the function f can not be conjugate to any function of bounded variation. □

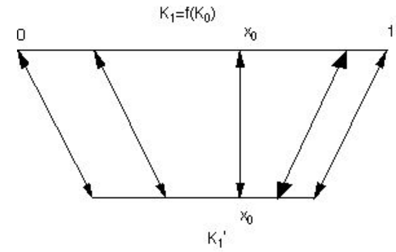
Proof of Theorem 2.2.b . We start by claiming that there exists a continuous function $g : [0,1] \to [0,1]$ such that $g(0) = 0, g(1) = 1,$ and $s_g(y) = \infty$ for all $y \in [0,1]$. In fact, a nice construction of such a function has been given by Foran; we refer to [Br] pages 148-150 for further details. Now define $\hat{g} = g(1-2|x-\frac{1}{2}|)$ and let

$$f(x) = 2^{-n}\hat{g}(2^n(x-2^{-n})),\ x \in (2^{-n}, 2^{-n+1}],\ n = 1,2,\dots,$$

$$f(0) = 0.$$

This gives us a wavelike function whose ripples are shrinking copies of the function $\hat{g}$. The function f is continuous, stable with $\mathrm{diam}(f^n([0,1])) = 2^{-n}$ and $s_f(y) = \infty$ for all $y \in [0,\frac{1}{2}]$. We have thus constructed a stable function satisfying the conditions in Theorem 2.2.a. □

Figure 1:



Proof of Theorem 2.4. We introduce

$$|D|f(x) = \limsup_{\Delta \to 0} \left| \frac{f(x+\Delta x) - f(x)}{\Delta x} \right|.$$

It can be checked that the following assertion is true: $|f(x)-f(y)| \leq M|x-y|$ for all $x, y \in [0,1]$ if and only if $|D|f(x) \leq M$ for all $x \in [0,1]$.

We assume that $M \geq 1$ since otherwise there is nothing to prove. Introduce $K_0 = [0,1]$ and $K_n = f^n(K_0)$, where f^n is f iterated n times. Then, since f is a Lipschitz function and thereby continuous, $\{K_n\}, n \geq 0$, forms a nested decreasing sequence of compact intervals. Due to the stability of f these intervals K_n shrink to a point, which we denote by x_0. Evidently x_0 is the fixed point of f. Again, due to the stability of f, $K_{n+1} \neq K_n$ for all n or $K_n = \{x_0\}$ for some n.

Now we shrink $K_1 = f(K_0)$ linearly by the factor $\frac{1}{2M}$ and centre at x_0, that is, we apply the linear function $\frac{1}{2M}(x-x_0)+x_0$. We get some interval K_1' as the image of K_1 under this linear transformation. After this we translate the (possibly) two intervals of $K_0 \setminus K_1$ towards the interior of K_0 until they reach K_1' in the sense that each of them gets a common endpoint with K_1' (see Fig.1).

The resulting action defined on the interval K_0 is a continuous one-to-one transformation of K_0 into itself, which we denote by a function h_1. Let $f_1 = h_1 \circ f \circ h_1^{-1}$ and $K_0' = h_1(K_0)$. K_0' consists of K_1' and the two translated intervals.

Evidently $|D|f_1 \leq \frac{M}{2M} = \frac{1}{2}$ on the translated intervals and on K_1' we again have $|D|f_1 \leq M$. Note also that $\{f, K_0\}$ is conjugate to $\{f_1, K_0'\}$. Now we apply the same procedure to the interval K_0' but we now shrink the interval $f_1(K_1')$ by the factor $\frac{1}{2M}$ (linearly, leaving the point x_0 fixed) which gives an interval K_2', and translate the (possibly) two intervals of $K_0' \setminus f_1(K_1')$ until they reach K_2'.

If we repeat this procedure infinitely many times we get some interval Δ (as the limit of the nested intervals $K_0, K_0', \dots$) and some functions g (as the limit of the functions $f, f_1, \dots$), for which $|D|g \le \frac{1}{2}$ on Δ, such that $\{g, \Delta\}$ is conjugate to $\{f, K_0\}$.

The interval Δ has positive length since it is the union of all the translated intervals and $\{x_0\}$. Observe that if $K_n = \{x_0\}$ for some n the process stops after a finite number of steps.

Finally, by expanding (linearly) the interval Δ to the interval $[0,1]$, we translate the function g to the interval $[0,1]$, preserving its contractivity factor $\frac{1}{2}$. Theorem 2.4 is proved. □

Proof of Theorem 2.5. Let all definitions be as in the proof of Theorem 2.4. Let M_n be the Lipschitz constant of f on $[0,1] \setminus K_n$. Now proceed exactly as in the proof of Theorem 2.4 but instead of always shrinking by a constant factor shrink the interval K_n by a factor $\frac{1}{2M_n}$. The rest of the proof goes through as in the proof of Theorem 2.4. □

Proof of Theorem 2.8. For concreteness suppose that f is an increasing function on $[0,1]$ with the unique fixed point x_0 inside the open interval $(0,1)$. We will show how to conjugate this function to the linear function $L(x) = \frac{1}{2}(x - \frac{1}{2}) + \frac{1}{2}$ (having $\frac{1}{2}$ as fixed point) defined on the interval $[0,1]$. Observe that $L([0,1]) = [\frac{1}{4}, \frac{3}{4}]$.

Let h be any homeomorphism mapping the half-open interval $[0, f(0))$ onto $[0, \frac{1}{4})$ and $(f(1), 1]$ onto $(\frac{3}{4}, 1]$. We can prove that for an arbitrary point $x \in [0,1]$, $x \ne x_0$, there exists a unique non-negative integer n (depending on x), such that $f^{-n}(x) \in [0, f(0)) \cup (f(1), 1]$. Here $f^{-n}(x)$ denotes the nth iterate of $f^{-1}(x)$ if $n > 0$ and $f^{-0} = f$.

Now we extend h to the whole interval $[0,1]$ by putting $h(x) = L^n \circ h \circ f^{-n}(x)$, if $f(0) \le x \le f(1)$, $x \ne x_0$ and $h(x_0) = \frac{1}{2}$; here L^n is L iterated n times. It is not difficult to check that h is a homeomorphism of $[0,1]$ onto itself and that f is conjugate to the linear function L via h. Theorem 2.8 is proved. □

Proof of Proposition 2.9. Let K be connected and $\{\mathcal{F}, K\}$ a locally stable IFS. Let us fix $F \subset \mathcal{F}^\infty$. By the assumptions we can, for each point $x \in K$, find a neighbourhood $U(x)$ of x (in the topology induced by K) such that $\operatorname{diam}(F_n(U(x))) \to 0$ as $n \to 0$. Due to a standard argument about finite coverings we can choose a finite number of points $x_1, x_2, \dots, x_n$ such that $\{U(x_j)\}_{j=1}^n$ covers K.

By using the triangle inequality we conclude that if, for some indices $i \ne j$, the sets $U(x_i)$ and $U(x_j)$ have non-empty intersection, then

$$\operatorname{diam}(F_n(U(x_i) \cup U(x_j))) \to 0 \text{ as } n \to \infty.$$

Finally, due to the connectedness of K we conclude that the set $\bigcup_{j=1}^{n} U(x_j)$ is also connected. From this the assertion of Proposition 2.9 follows. □

Proof of Proposition 2.10. Assume that the system consists of a finite number of functions and that it is stable but not uniformly stable. This means that there exists a positive number $a > 0$ and an infinite sequence of elements $F^{(j)} \in \mathcal{F}^\infty$, $j = 1, 2, \dots$, and an increasing sequence of natural numbers $n_1, n_2, \dots$, such that

$$\mathrm{diam}(F^{(j)}_{n_j}(K)) > a, \ j = 1, 2, \dots .$$

Now let $g_1, g_2, \dots, g_n, \dots$ be an infinite sequence of functions in $\mathcal{F}$ such that for any natural number n the sequence $g_1, g_2, \dots, g_n$ appears as a prefix (first initial component) for infinitely many elements $F^{(j)}$. Such a sequence exists due to the finiteness of the set $\mathcal{F}$ (first we choose g_1, then g_2 and so on). For any natural n we now choose an element $F^{(j)}$, starting with $g_1, g_2, \dots, g_n$, so that $n_j > n$. We get that

$$a < \mathrm{diam}(F^{(j)}_{n_j}(K)) \le \mathrm{diam}(F^{(j)}_{n}(K)) = g_1 \circ g_2 \circ \dots \circ g_n(K).$$

Introduce $F = (g_1, g_2, \dots, g_n, \dots) \in \mathcal{F}^\infty$. Then the last inequality gives that $\mathrm{diam} F_n(K) > a$ for any positive n. This contradicts the assumption that the system Proposition 2.10 is stable. □

Proof of Proposition 2.12. Let $\{\mathcal{F}', K'\}$ be an IFS conjugate to a stable system $\{\mathcal{F}, K\}$ in the sense of Definition 1.4. Let $F = (f_1, f_2, \dots) \in \mathcal{F}^\infty$. Consider $F' = (f'_1, f'_2, \dots) \in (\mathcal{F}')^\infty$, where $f'_j = h \circ f_j \circ h^{-1}$, $j = 1, 2, \dots$. Then $F'_n(K') = f'_1 \circ f'_2 \circ \dots \circ f'_n(K') = h \circ f_1 \circ f_2 \circ \dots \circ f_n \circ h^{-1}(K') = h \circ f_1 \circ f_2 \circ \dots \circ f_n(K) = h(F_n(K))$. The sets $F_n(K)$, $n \ge 1$, shrink to a point $x_0 \in K$ and the assertion of Proposition 2.12 follows from the continuity of h at the point x_0. For the uniform stability respectively we use the uniform continuity of functions continuous on compact sets. □

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