Even cycle decompositions of 4-regular graphs and line graphs

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Abstract. An even cycle decomposition of a graph is a partition of its edge into even cycles. We first give some results on the existence of even cycle decomposition in general 4-regular graphs, showing that K_5 is not the only graph in this class without such a decomposition.

Motivated by connections to the cycle double cover conjecture we go on to consider even cycle decompositions of line graphs of 2-connected cubic graphs. We conjecture that in this class even cycle decompositions always exists and prove the conjecture for cubic graphs with oddness at most 2. We also discuss even cycle double covers of cubic graphs.

1. Introduction

One of the first theorems we learn in graph theory is that a graph has a cycle decomposition if and only if it is eulerian. Here a cycle decomposition of a graph G is a partition of its edge set where each part is a cycle in G. The proof of this theorem is very simple but when additional constraints are imposed on the structure of the cycle decomposition numerous connections to some of the hardest problems in graph theory appear. Two enjoyable surveys can be found in [Jac93] and [Fle01].

Perhaps the simplest additional constraint is to require that all cycles in the decomposition have even length, here called an even cycle decomposition or ECD. An obvious necessary condition for this be possible is that each 2-connected component, called a block, of G has an even number of edges. In 1981 Seymour [Sey81] proved that an eulerian planar graph , in which every block has an even number of edges, has an ECD. In 1994 Zhang [Zha94] strengthened this result by replacing planarity with the condition that G has no K_5-minor. Zhang also conjectured that K_5 is the only 3-connected eulerian graph without an ECD, but an infinite family of counterexample was later found by Jackson [Jac93]. Jackson in turn asked whether K_5 was the only 4-connected such example. This question was answered by Rizzi [Riz01] who constructed an infinite family of 4-connected graphs with vertices of degree 4 and 6, and no ECD.

The aim of this paper is to consider the existence of ECDs in 4-regular graphs. As we show in the next section the restriction to regular graphs is not enough to make K_5 the only ECD-free such graph in the class under consideration. However we conjecture that 4-regular line graphs of 2-connected cubic graph have ECDs. We next discuss how this conjecture relates to even cycle double covers of cubic graphs, and cubic graphs having such covers. Finally we prove the conjecture for line graphs of cubic graphs of oddness at most 2.

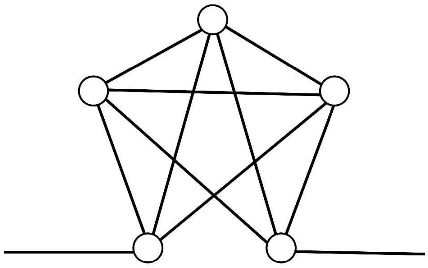

FIGURE 1. The edge gadget

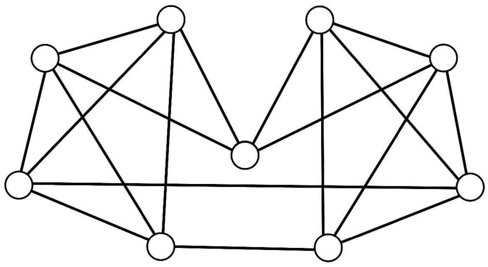

FIGURE 2. A 3-connected 4-regular graph with no even cycle decomposition

2. General 4-regular graphs

For a 4-regular graph any 2-connected component must have an even number of edges, and the simplest of the conditions necessary for the existence of and ECD is always met if the graph has connectivity at least 2.

As mentioned in the introduction the construction of Rizzi, and that of Jackson, do not lead to 4-regular graphs. However for 2-connected graphs it is easy to construct infinitely many graphs without an ECD. If an edge of a 2-connected 4-regular graph is replaced by the gadget in Figure 1 the resulting graph will not have an ECD.

In Figure 2 we give an example of a 3-connected, and 4-edge-connected, graph which does not have a ECD. We do not have a short proof demonstrating that the graph does not have an ECD, only a case analysis, but since the graph is so small the reader could alternatively verify the claim by computer.

A natural question is whether a typical 4-regular graph on n vertices has en ECD. Zhang's result [Zha94] does not tell us anything here since almost all 4-regular graphs have a K_5-minor [Mar04], and in fact much larger complete minors [FKO09]. The following theorem settles the question for even n:

Theorem 2.1 ([KW01])**.** *A random 4-regular graph asymptotically almost surely decomposes into two hamiltonian cycles.*

For odd n this is not helpful for our purposes, however we conjecture

Conjecture 2.2. *A random 4-regular graph on $2n+1$ vertices asymptotically almost surely has a decomposition into a C_{2n} and two other even cycles.*

Note that the two shorter even cycles must intersect in exactly one vertex.

A question which we have not managed to settle is

Problem 2.3. *Are there 3-connected 4-regular graphs with girth at least 4 which do not have an ECD?*

3. Line graphs of cubic graphs

A class of 4-regular graphs with interesting structural properties are the line graphs of cubic graphs. In particular they have strong connections to cycle covers of cubic graphs, as discussed in [Jac93, Fle01], and that was one of our motivations for the current work. Given a graph G let $L(G)$ denote its line graph.

Our main conjecture is

Conjecture 3.1. *If G is a 2-connected cubic graph then $L(G)$ has an ECD.*

We have not been able to prove this conjecture, but as we shall demonstrate counter-examples, should they exit, must be rare.

3.1. Even cycle double covers. Recall that a cycle double cover of a graph G is a family of cycles from G such that every edge of G belongs to exactly two of the cycles. The well known cycle double cover conjecture claims that all 2-connected cubic graphs have a cycle double cover.

A simple observation is that

Lemma 3.2. *A three edge-colourable cubic graph has a cycle double cover. In addition there is such a cover in which every cycle is even.*

We will refer to a cycle double cover containing only even length cycles as an even cycle double cover, or a ECDC.

Lemma 3.3. *If a cubic graph G has an even cycle double cover then $L(G)$ has an even cycle decomposition.*

Proof. Assume that the cycle double cover consists of the cycles $C_1, \dots, C_k$. Let $e_{i,1}, \dots, e_{i,n_i}$ be the edges of C_i in the order they appear in C_i. Now i we view $e_{i,1}, \dots, e_{i,n_i}$ as vertices in $L(G)$ they define a cycle $\tilde{C}_i$ in $L(G)$ as well, of the same length as C_i, and since each edge of G belongs to two such cycles each vertex in $L(G)$ will lie in exactly two of the cycles.

Finally, since two cycles C_1 and C_2 of a cycle double cover in a cubic graph, cannot intersect in two incident edges, every edge of $L(G)$ must belong to exactly one of the cycles $\tilde{C}_i$. Hence $\tilde{C}_1, \dots, \tilde{C}_k$ is an ECD of $L(G)$ □

Together the lemmata give us

Theorem 3.4. *If G is a three edge-colourable cubic graph then $L(G)$ has an even cycle decomposition*

It is an immediate consequence of the configuration model for random regular graphs [Bol80] that almost all cubic graphs are three edge-colourable. We s ay that a property holds asymptotically almost surely if the probability that a gaph on n vertices has the property tends to 1 as $n \to \infty$.

Corollary 3.5. *If G is a random cubic graph then asymptotically almost surely $L(G)$ has an ECD.*

We are now led to ask: which cubic graphs have even cycle double covers? In Szekeres's first paper on the cycle double cover conjecture [Sze73] he pointed out that the Petersen graph does not have an ECDC, and also claimed to prove that in fact a cubic graph has an ECDC if and only if it three edge-colourable. However Preissman later pointed out [Pre80] the proof in incorrect, and showed that there is an infinite family of snarks with ECDCs.

We say that a cubic graph G is a 2-sum of two cubic graphs G_1 and G_2 if there exists an edge cut of size two in G such that if we delete the edges of the cut we are left with two graphs G_1' and G_2' which are formed by deleting an edge from G_1 and G_2 respectively. We say that G is a 3-sum of G_1 and G_2 if there exists an edge cut of size three in G such that if we delete the edges of the cut we are left with two graphs G_1'' and G_2'' which are formed by deleting a vertex from G_1 and G_2 respectively.

Starting with the Petersen graph it is easy to construct infinitely many cubic graphs without an ECDC by taking 2-sums or 3-sums with bipartite cubic graphs. However it is easy to check the standard connectivity and girth reductions for snarks introduced by Isaacs [Isa75], see also [Jae85], have the following properties

Lemma 3.6.

(1) Assume that the cubic graph G is a 2-sum or a 3-sum of two graphs G_1 and G_2. If G does not have an ECDC then at least one of G_1 and G_2 does not have an ECDC.

(2) If the cubic graph G does not have an ECDC and G contains a triangle then the graph obtained by contracting the triangle to a single vertex does not have an ECDC.

(3) If the cubic graph G does not have an ECDC and G contains a C_4 then we can construct a smaller graph with no ECDC by deleting the C_4 and adding two edges to form a new cubic graph.

With this lemma in mind we see that the study of even cycle double covers can be focused on snarks. We have used a computer to search for ECDCs of the small snarks, which can be downloaded from [Roy]. We found at least one ECDC in all snarks but the Petersen graph.

Observation 3.7. *The only snark on $n \leq 28$ vertices which does not have an ECDC is the Petersen graph.*

It is natural to ask

Problem 3.8. *Is the Petersen graph the only snark which does not have an even cycle double cover?*

3.2. Line graphs of cubic graph with larger oddness. As we have seen Conjecture 3.1 is true for three edge-colourable graph. One way of quantifying how far a cubic graph is from being three edge-colourable is by its *oddness*

Definition 3.9. A 2-connected cubic graph G has oddness $o(G) = k$ if k is the smallest number of odd cycles in a 2-factor of G.

By Petersen's theorem [Pet91] every 2-connected cubic graph has at least three 2-factors. A three edge-colourable graph has oddness 0, since the edges of the first to colours induce a bipartite 2-factor. The Petersen graph P has $o(P) = 2$.

Results in terms of oddness have been studied for cycle double covers. Huck and Kochol [HK95] proved that cubic graphs of oddness 2 have cycle double covers, and later this was extended by Häggkvist and McGuinness [HM05] and [Huc01] to oddness 4.

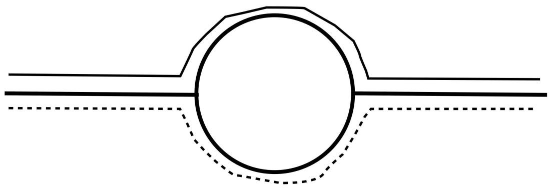

FIGURE 3. A simple intersection

Our final result is

Theorem 3.10. *If G is a 2-connected cubic graph with $o(G) = 2$ then $L(G)$ has an ECD.*

Proof. Let $\mathcal{C} = \{C_1, \dots, C_k\}$ be a 2-factor of G with only two odd cycles, where C_1 and C_2 are the odd cycles.

Since G is 2-connected we can find two vertex disjoint paths P_1 and P_2, each with exactly one vertex in C_1 and one in C_2. We also assume that P_1 and P_2 are shortest among all such paths. Let A_1 and A_2 be the two edge-disjoint paths in C_1 joining the endpoints of P_1 and P_2. Since C_1 is odd exactly one of A_1 and A_2 mus have odd length, we may assume that it is A_1. Let p_1 be the path formed by A_1 and the first edge of P_1 and P_2. Let p_2 be form by A_2 and the same edges from P_1 and P_2.

We will now use p_1 and p_2 to construct a covering, by paths and cycles, of P_1 and P_2 such that the edges are covered twice if they belong to $\mathcal{C} \setminus (P_1 \cup P_2)$, once if they belong to $\mathcal{C}$, and each cycle in the covering is even. We will do this by following both paths from C_1 to C_2 in parallel and extend the graphs p_1 and p_2 appropriately. In all the following figures we imagine that the path are going left to right towards C_2.

If only one of P_1 and P_2 intersect a cycle $C_i, i \geq 3$ from $\mathcal{C}$ then we route two two paths p_1 and p_2 through C as shown in Figure 3

If both P_1 and P_2 intersect $C_i, i \geq 3$ then there are two possible configurations, shown in figures 4 and 5. In the situation depicted in Figure 4, both p_1 and p_2 are routed past C_i, and since C_i has even length exactly one of them will have odd length after doing so, even though which one might have changed.

In the situation depicted in Figure 5 one of the incoming paths p_1 and p_2 is closed to form a cycle. If the left path from u to v in C_i has odd length we choose to close the one of the p_i:s which has odd length, otherwise we close the even one, thereby forming an even cycle. In those situation we then continue to the right with a new path in place of the one we close. Since C_i has even length exactly one of the two continuing paths will have odd length.

When two paths p_1 and p_2 reach C_2 we use the two paths in C_2 between the endpoint of P_1 and P_2 to closed them off into cycles. Since C_2 has odd length, exactly one of the two paths within C_2 have odd length so we can ensure that both the cycles now formed are even. Call this family of cycles D_0.

Given a cycle C in G we say that there are two cycles in $L(G)$ associated with C. One cycle C' whose vertices are consecutive edges in C, and one cycle C'' whose vertices are

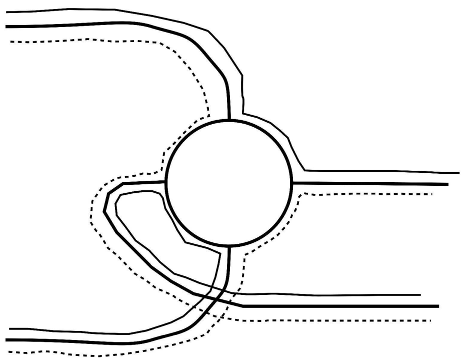

FIGURE 4. The first kind of double intersection

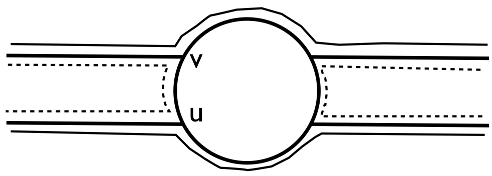

FIGURE 5. The second kind of double intersection

alternatingly edges incident with with C, but not in C, and edges in C. An example is shown in Figure 6. Note that C'' is twice as long as C and hence always an even cycle.

We are now ready to construct the ECD of $L(G)$. Given the 2-factor $\mathcal{C}$ we get a cycle decomposition D_1 by taking the associated cycles for all cycles in $\mathcal{C}$, however C'_1 and C'_2 are odd. We know delete C'_i from D_1, for all i, to form D_2. At each edge of $(P_1 \cup P_2) \setminus E(\mathcal{C})$ we modify each C'' in D_2 to instead use an edge of a C'. This does not change the parity of the cycle lengths, since each P_i is incident with two or zero such edges in C. This gives us a collection D_3 of even cycles. Finally we form our ECD D by including all cycles C' for $C \in D_0$. □

All snarks on $n \leq 28$ vertices have oddness 2, again by computational observation using the snarks from [Roy], and as far as we know the size of smallest snark with oddness 2 has not been determined. Note that the cycle decomposition constructed in Theorem 3.10 does not come from and ECDC, so for the small snarks, other than the Petersen graph, there exist at least two distinct even cycle decompositions. We believe that a more extensive case analysis would make it possible to prove the conjecture for oddness 4 as well.

Regarding the oddness of random cubic graph we have made the following conjecture.

Conjecture 3.11. *Asymptotically almost surely a cubic graph G with $o(g) > 2k$ $g \geq 0$ has $o(G) = 2k + 2$.*

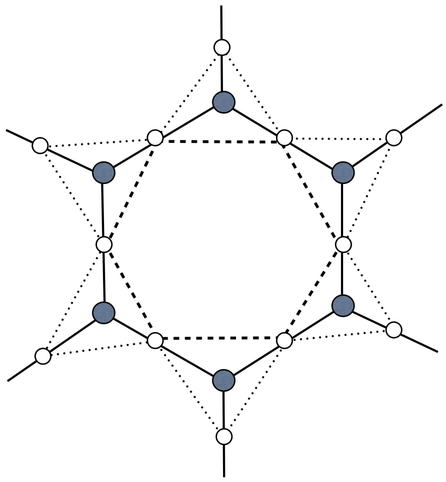

FIGURE 6. The cycles associated with C for a 6-cycle. C' with dashed edges and C'' with dotted.

Even for $k = 0$ the conjecture seems challenging.

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