On the density of 2-colourable 3-graphs in which any four points span at most two edges

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Abstract

Let $\mathrm{ex}_2(n, \mathcal{K}_4^-)$ be the maximum number of edges in a 2-colourable $\mathcal{K}_4^-$-free 3-graph (where $\mathcal{K}_4^- = \{123, 124, 134\}$). The 2-chromatic Turán density of $\mathcal{K}_4^-$ is $\pi_2(\mathcal{K}_4^-) = \lim_{n\to\infty} \mathrm{ex}_2(n, \mathcal{K}_4^-)/\binom{n}{3}$. We improve the previously best known lower and upper bounds of 0.25682 and $3/10 - \epsilon$ respectively by showing that

$$0.2572049 \leq \pi_2(\mathcal{K}_4^-) < 0.291.$$

This implies the following new upper bound for the Turán density of $\mathcal{K}_4^-$

$$\pi(\mathcal{K}_4^-) \leq 0.32908.$$

In order to establish these results we use a combination of the properties of computer generated extremal 3-graphs for small n and an argument based on "super-saturation". Our computer results determine the exact values of $\mathrm{ex}(n, \mathcal{K}_4^-)$ for $n \leq 19$ and $\mathrm{ex}_2(n, \mathcal{K}_4^-)$ for $n \leq 17$, as well as the sets of extremal 3-graphs for those n.

1 Definitions

A 3-graph $\mathcal{F}$ of order $n \geq 1$ consists of a vertex set V of size n and a collection of unordered triples from V called *edges.* If $\mathcal{F}$ and $\mathcal{H}$ are 3-graphs then $\mathcal{H}$ is said to be *$\mathcal{F}$-free* if it contains no isomorphic copy of $\mathcal{F}$. The maximum number of edges in an F-free 3-graph of order n is denoted by $\mathrm{ex}(n, \mathcal{F})$. Determining $\mathrm{ex}(n, \mathcal{F})$ is known as the Turán problem for $\mathcal{F}$. The smallest 3-graph for which the associated Turán problem is non-trivial is the unique 3-graph of order 4 with 3 edges: $\mathcal{K}_4^- = \{123, 124, 134\}$. Note that a 3-graph $\mathcal{H}$ is $\mathcal{K}_4^-$-free if and only if no four vertices in $\mathcal{H}$ span more than two edges.

Determining $\mathrm{ex}(n, \mathcal{K}_4^-)$ is a well studied open problem. Since an exact solution seems very hard to find (unless n is small) we may instead consider the problem of determining the *Turán density*

$$\pi(\mathcal{K}_4^-) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{K}_4^-)}{\binom{n}{3}}.$$

This Turán problem can also be viewed as a question concerning *double covering designs*. An (n,k,t)-*covering design* is a family $\mathcal{D}$ of k-sets from an n-set V with the property that every subset of V of size t is contained in at least one k-set from $\mathcal{D}$. An (n,k,t)-covering design with the property that every t-set from V is contained in at least two k-sets from $\mathcal{D}$ is called a *double covering design*.

If $\mathcal{H}$ is a $\mathcal{K}_4^-$-free 3-graph then $\mathcal{D} = \{[n]\backslash e \mid e \in \binom{[n]}{3}\backslash\mathcal{H}\}$ is an $(n, n-3, n-4)$-double covering design. Indeed if $\mathcal{H}$ is extremal (i.e. $|\mathcal{H}| = \mathrm{ex}(n,\mathcal{K}_4^-)$) then $\mathcal{D}$ is *optimal* in the sense that no other $(n, n-4, n-3)$-double covering design is smaller.

It was shown in [Tal07] that the problem of determining $\pi(\mathcal{K}_4^-)$ is related to the following so-called *chromatic Turán problem*.

A 3-graph is said to be *k-colourable* if there is a partition $V = A_1\dot\cup A_2\dot\cup\cdots\dot\cup A_k$ of its vertices so that none of the vertex classes A_i contains an edge. We denote the maximum number of edges in a k-colourable $\mathcal{K}_4^-$-free 3-graph of order n by $\mathrm{ex}_k(n,\mathcal{K}_4^-)$. The corresponding k-chromatic Turán density is $\pi_k(\mathcal{K}_4^-) = \lim_{n\to\infty} \mathrm{ex}_k(n,\mathcal{K}_4^-)/\binom{n}{3}$.

Our main result is the following improvement in lower and upper bounds for the 2-chromatic Turán density $\pi_2(\mathcal{K}_4^-)$.

Theorem 1 *The 2-chromatic Turán density $\pi_2(\mathcal{K}_4^-)$ satisfies*

$$0.2571912 \leq \pi_2(\mathcal{K}_4^-) < 0.291.$$

The lower bound follows from a construction, given in Section 3, while the upper bound follows from a combination of a computational result, giving $\mathrm{ex}_2(16,\mathcal{K}_4^-)$, and the "super-saturation" method.

An immediate corollary of this result is the following improved upper bound for the Turán density of $\mathcal{K}_4^-$.

Corollary 2 *The Turán density of $\mathcal{K}_4^-$ satisfies*

$$\pi(\mathcal{K}_4^-) < 0.32908.$$

This result follows simply from Theorem 1 using the calculations in [Tal07].

2 Extremal 3-graphs for the 2-chromatic and general Turán problems

For small n it is possible to find the complete set of extremal $\mathcal{K}_4^-$-free 3-graphs computationally, and thereby the value of $\mathrm{ex}(n,\mathcal{K}_4^-)$ as well. Earlier this had been done by a direct combinatorial search method for $n \leq 12$ in [LvRSW06], and we will here extend this to all $n \leq 19$ and give an improved bound for $\text{ex}(20, \mathcal{K}_4^-)$.

The basic idea underlying our computation is the following simple lemma, established by considering the average degree of the 3-graph.

Lemma 3 *If G_1 is an $\mathcal{K}_4^-$-free 3-graph on n vertices and m edges then there exists an $\mathcal{K}_4^-$-free 3-graph G_2 on $n-1$ vertices and at least $m - \left\lfloor \frac{3m}{n} \right\rfloor$ edges, such that $G_2 = G_1 \setminus v$, for some $v \in V(G_1)$.*

This lemma tells us that the size of an extremal 3-graph on $n+1$ vertices can be bounded in terms of $\text{ex}(n, \mathcal{K}_4^-)$. Furthermore, if we have found all $\mathcal{K}_4^-$-free 3-graphs on n vertices with e edges, where $m - \lfloor 3m/n \rfloor \leq e \leq m$, then we can construct all $\mathcal{K}_4^-$-free 3-graphs on $n+1$ vertices and m edges as follows:

1. Let S be the set of all $\mathcal{K}_4^-$-free 3-graphs on n vertices with e edges , where $m - \left\lfloor \frac{3m}{n} \right\rfloor \leq e \leq m$.
2. Given a 3-graph $G \in S$ let U_G be the set of all $\mathcal{K}_4^-$-free 3-graphs which can be constructed from G by adding a new vertex v to $V(G)$ and a set of $m - |E(G)|$ edges containing v.
3. Let $U = \cup_G U_G$ and let S' be the set of non-isomorphic 3-graphs in U.
4. S' is the set of all $\mathcal{K}_4^-$-free 3-graphs on n vertices and m edges.

That this simple procedure works follows directly from Lemma 3. If step 2 wer done by a brute force search this procedure would be too slow for large n. Instead we formulated the extension step as an integer programming problem which was then solved using the integer programming solver included in GNU's glpk-package [Mak]. Finally the isomorphism reduction in step 3 was done using Brendan McKay's Nauty [McK81]. The same procedure was used for creating the 2-chromatic extremal graphs, with the simple modification that the 3-graphs created in step 3 were required to be 2-chromatic and only 2-chromatic 3-graphs needed to be included in S.

The computational results are given in Figures 1 and 2. There are several interesting facts to note.

Let us recall that in [FF84] Frankl and Füredi gave a recursive construction by taking blow ups of the unique extremal $\mathcal{K}_4^-$-free 3-graph on 6 vertices. This provided a sequence of $\mathcal{K}_4^-$-free 3-graphs with asymptotic density $\frac{2}{7}$. From Figure 1 we see that for each $n \geq 11$ the unique extremal $\mathcal{K}_4^-$-free 3-graph of order n is this blow-up. An inspection of the 3-graphs with one edge less than the extremal one shows that for $n \geq 12$ all of these 3-graphs can be obtained by deleting an edge from the extremal 3-graph, except for a single additional 3-graph at $n = 15$. Similarly most, but not all, 3-graphs with two or three edges less than the extremal one can be obtained by deleting edges from the extremal 3-graph.

Mubayi [Mub03] had previously conjectured that the $\mathcal{K}_4^-$-free construction of Frankl and Füredi [FF84] was optimal for infinitely many values of n. Motivated by our computational results we give the following strengthening of this conjecture.

Conjecture 4 *For $n \geq 11$ the unique $\mathcal{K}_4^-$-free 3-graph of size $ex(n, \mathcal{K}_4^-)$ is the 3-graph constructed by the blow-up construction of [FF84].*

n	size	opt	opt−1	opt−2	opt−3
6	10	1	1	7	18
7	15	1	8	70	374
8	22	5	75	1308	15511
9	32	6	171	4426	91667
10	44	43	1343	41291	1139106
11	60	1	15	1058	53235
12	80	1	1	9	74
13	101	1	3	34	438
14	126	1	5	75	1062
15	156	1	5	54	758
16	190	1	6	79	1145
17	230	1	3	36	499
18	276	1	2	11	116
19	322	1	5		
20	<377				

Figure 1: Extremal and near-extremal $\mathcal{K}_4^-$-free 3-graphs..

The columns of Figure 1 are as follows: number of vertices; number of edges in the extremal $\mathcal{K}_4^-$-free 3-graphs; number of non-isomorphic extremal $\mathcal{K}_4^-$-free 3-graphs; number of non-isomorphic $\mathcal{K}_4^-$-free 3-graphs with respectively $1, 2$ and 3 edges less than the extremal $\mathcal{K}_4^-$-free 3-graphs.

For the 2-chromatic $\mathcal{K}_4^-$-free 3-graph we do not see the same type of stability as in the general problem. As Figure 2 shows the number of 3-graphs with size close to $\mathrm{ex}_2(n, \mathcal{K}_4^-)$ is much larger than for the general case and for most values of n the extremal 3-graph is not unique. Furthermore we note that the 3-graph used in the next section to give our lower bound for $\mathrm{ex}_2(n, \mathcal{K}_4^-)$, via the blow-up construction, has $n = 14$ vertices and only 114 edges, i.e. it is not the extremal 3-graph for that number of vertices.

n	size	opt	opt−1	opt−2	opt−3
7	14	3	36	307	1059
8	21	4	171	3470	39570
9	30	9	428	15182	359640
10	42	2	64	3549	146437
11	56	3	90	4113	182144
12	73	1	64	2424	108531
13	93	1	68	2734	110537
14	116	7	262	10901	
15	144	1	5	229	
16	174	7			
17	209-210				

Figure 2: 2-colourable extremal and near-extremal $\mathcal{K}_4^-$-free 3-graphs.

3 A new lower bound for the 2-chromatic Turán problem

Our lower bound is obtained by using the standard blow-up construction, see for example [FF89], starting with the 3-graph given at the end of this section. This is a 2-colourable 3-graph of order 14 with 114 edges and Lagrangian 0.042867489.... This value of the Lagrangian is given by the following, approximate, vector of weights for the vertices

$$W = (0.153354, 0.155487, 0.0296491, 0.109346, 0.105072, 0.0142802, 0.0140061, 0.0296513, 0.0368664, 0.141559, 0.0363688, 0.036865, 0.101128, 0.0363672) \quad (1)$$

We note that the vertices $\{1, 2, 4, 5, 10\}$ have much larger weights than the other vertices, and form an induced copy of the unique 2-colourable $\mathcal{K}_4^-$-free 3-graph on nine edges and six vertices.

Blowing this 3-graph up according to the vector W gives a sequence of 3-graphs with asymptotic density $0.2572049\ldots$, implying that $\pi_2 \geq 0.2572$.

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{{11, 13, 14}, {11, 12, 14}, {10, 11, 14}, {9, 12, 14}, {9, 12, 13},
{9, 11, 13}, {9, 10, 14}, {9, 10, 13}, {8, 12, 14}, {8, 12, 13}, {8,
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4, 10}, {3, 4, 6}, {2, 12, 14}, {2, 12, 13}, {2, 11, 13}, {2, 10, 14},
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{2, 10, 13}, {2, 7, 12}, {2, 7, 11}, {2, 7, 10}, {2, 6, 14}, {2, 6,
13}, {2, 5, 11}, {2, 5, 10}, {2, 5, 9}, {2, 5, 6}, {2, 4, 14}, {2, 4,
13}, {2, 4, 9}, {2, 4, 7}, {2, 3, 11}, {2, 3, 10}, {2, 3, 9}, {2, 3,
8}, {2, 3, 6}, {1, 11, 13}, {1, 11, 12}, {1, 10, 14}, {1, 10, 13}, {1,
10, 12}, {1, 9, 12}, {1, 8, 12}, {1, 7, 11}, {1, 7, 10}, {1, 6, 14},
{1, 6, 13}, {1, 6, 12}, {1, 5, 14}, {1, 5, 13}, {1, 5, 8}, {1, 5, 7},
{1, 4, 11}, {1, 4, 10}, {1, 4, 8}, {1, 4, 6}, {1, 3, 14}, {1, 3, 13},
{1, 3, 7}, {1, 2, 12}, {1, 2, 5}, {1, 2, 4}, {1, 2, 3}}.
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4 A new upper bound for the 2-chromatic Turán problem

For $\eta > 0$ let $n \geq n_0(\eta)$ be sufficiently large that $\mathrm{ex}_2(n, \mathcal{K}_4^-) \leq (\pi_2 + \eta)\binom{n}{3}$. Let $\mathcal{H}$ be a $\mathcal{K}_4^-$-free 2-colourable 3-graph of order n with $m = \mathrm{ex}_2(n, \mathcal{K}_4^-)$ edges, so $m \leq \pi_2'\binom{n}{3}$ (where $\pi_2' = \pi_2 + \eta$). To complete the proof of Theorem 1 it is sufficient to show that $\pi_2' < 0.291$.

Let $V(\mathcal{H}) = A\dot{\cup}B$ be a 2-colouring of $\mathcal{H}$ and suppose that $|A| = \alpha n$, for some $1/2 \leq \alpha \leq 1$ (that is we take A to be the larger of the two vertex classes). Let βm be the number of edges of $\mathcal{H}$ that meet A in two vertices, so $0 \leq \beta \leq 1$.

For $C \subseteq V$ let $e(C)$ denote the number of edges of $\mathcal{H}$ contained in C. For $0 \leq i \leq 4$ let $q_i = \#\{C \in V^{(4)} : e(C) = i\}$ and write $q_1 = \mu m n$.

Lemma 5 *If α, β, μ and π_2' are as above and $\alpha = \frac{1+\epsilon}{2}$, $\beta = \frac{1+\delta}{2}$ then*

$$\pi_2' \leq \frac{3(1-\mu)(1-\epsilon^2)^2}{(10 - 6\epsilon^2 - 8\epsilon\delta + 2\delta^2 + 2\epsilon^2\delta^2)} + O(n^{-1}).$$

Proof: Counting edges in 4-sets we have

$$m(n-3) = q_1 + 2q_2.$$

Denoting the degree of a pair of vertices by

$$d_{xy} = \#\{z \in V : xyz \in \mathcal{H}\}.$$

and using the fact that

$$\sum_{xy \in V^{(2)}} \binom{d_{xy}}{2} = q_2$$

we obtain

$$mn = q_1 + \sum_{xy \in V^{(2)}} d_{xy}^2.$$

Hence by considering pairs of vertices from $A^{(2)}, B^{(2)}$ and $A \times B$, and using the convexity of x^2 we have

$$(1-\mu)mn \geq \frac{(\beta m)^2}{\binom{\alpha n}{2}} + \frac{((1-\beta)m)^2}{\binom{(1-\alpha)n}{2}} + \frac{4m^2}{\alpha(1-\alpha)n^2}.$$

Let $\alpha = (1+\epsilon)/2$ and $\beta = (1+\delta)/2$, so $0 \leq \epsilon \leq 1$ and $-1 \leq \delta \leq 1$. Using $m = \pi_2'\binom{n}{3}$ and rearranging we obtain

$$\pi_2' \leq \frac{3(1-\mu)(1-\epsilon^2)^2}{(10 - 6\epsilon^2 - 8\epsilon\delta + 2\delta^2 + 2\epsilon^2\delta^2)} + O(n^{-1}).$$

□

A similar argument also establishes the following simpler upper bound.

Lemma 6 *If α and π_2' are as above and $\alpha = \frac{1+\epsilon}{2}$ then*

$$\pi_2' \leq \frac{3}{\frac{4}{1-\epsilon^4} + \frac{6}{1-\epsilon^2}}.$$

We now require a result of Frankl and Füredi characterising 3-graphs in which any 4-set spans exactly 0 or 2 edges. In order to describe their result we need two constructions.

Let $\mathcal{S}$ be the following 3-graph of order 6 with 10 edges

$$\mathcal{S} = \{123, 124, 345, 346, 156, 256, 135, 146, 236, 246\}.$$

Let $|V| = n$ and suppose that V is partitioned as $V = V_1 \cup \cdots \cup V_6$. For such a partition we define $\mathcal{G}_\mathcal{S}$ to be the "blow-up" of $\mathcal{S}$. So $\mathcal{G}_\mathcal{S}$ has vertex set V and edge set

$$\mathcal{G}_\mathcal{S} = \{v_{i_1}v_{i_2}v_{i_3} : 1 \leq i_1 < i_2 < i_3 \leq 6,\ i_1i_2i_3 \in \mathcal{S} \text{ and } v_{i_j} \in V_{i_j}\}.$$

Let $\mathcal{P}$ be an arrangement of n points on the unit circle with the property that no line joining two points passes through the origin. We define $\mathcal{G}_\mathcal{P}$ to be the 3-graph with vertex set $\mathcal{P}$ and an edge for each triple uvw such that the corresponding triangle contains the origin.

Theorem 7 (Frankl and Füredi [FF84]) *If $\mathcal{G}$ is a 3-graph of order n in which every four points span exactly 0 or 2 edges then $\mathcal{G}$ is isomorphic to either $\mathcal{G}_\mathcal{S}$ or $\mathcal{G}_\mathcal{P}$ (for some vertex partition $V = V_1 \cup \cdots \cup V_6$ or arrangement of points $\mathcal{P}$ respectively).*

Corollary 8 *If $\mathcal{G}_2$ is a 2-colourable 3-graph of order n in which every four points span exactly 0 or 2 edges then $\mathcal{G}_2$ is isomorphic to $\mathcal{G}_\mathcal{P}$ for some arrangement of points on the unit circle $\mathcal{P}$. Furthermore if $n = 2k$ then*

$$e(\mathcal{G}_2) \leq 2\binom{k+1}{3}.$$

Proof: The fact that $\mathcal{G}_2$ is isomorphic to $\mathcal{G}_\mathcal{P}$ follows trivially from Theorem 7 since $\mathcal{G}_\mathcal{S}$ is not 2-colourable (as $\mathcal{S}$ is not 2-colourable).

The bound on the number of edges in $\mathcal{G}_2$ is given in the original paper [FF84]. They show that to maximize the number of edges in $\mathcal{G}_{\mathcal{P}}$ (for a fixed number of points n) we may form $\mathcal{P}$ by taking a regular $(2j+1)$-gon and placing d_i points at each of its vertices, in such a way that $d_1, \ldots, d_{2j+1}$ are as equal as possible. This is then maximized by taking j to be as large as possible (subject to the condition $2j+1 \leq n$). Thus for $n = 2k$ the maximum number of edges in a 3-graph $\mathcal{G}_{\mathcal{P}}$ of order n is given by taking a $(2k-1)$-gon and placing a single point at each of its vertices except one, at which two points are placed. The number of edges this gives equals the bound $2\binom{k+1}{3}$. □

We say that a 2-colourable 3-graph $\mathcal{G}$ is *balanced* if there is a partition $V(\mathcal{G}) = U \dot\cup W$ with $|U| = |W|$ and none of the edges of $\mathcal{G}$ lie in U or W. For a 3-graph $\mathcal{G}$ and an even integer n we define $\mathrm{ex}_B(n, \mathcal{G})$ to be the maximum number of edges in a balanced $\mathcal{G}$-free 3-graph.

We now require the following computational result.

Lemma 9 *If $\mathcal{G}$ is a 2-colourable $\mathcal{K}_4^-$-free 3-graph of order 16 then $\mathcal{G}$ contains at most 174 edges. Moreover if $\mathcal{G}$ is balanced then it contains at most 173 edges, i.e. $ex_2(16, \mathcal{K}_4^-) = 174$ and $ex_B(16, k) = 173$.*

Proof: By computation. □

We will say that a set $D \subset V(\mathcal{H})$ is *good* if it contains a 4-set which itself contains exactly one edge, otherwise we say that D is *bad*. For $k \geq 1$ let

$$\mathcal{C}_k = \{C \in V^{(2k)} : |C \cap A| = |C \cap B| = k\}.$$

Corollary 8 implies that if $C \in \mathcal{C}_k$ is bad then $e(C) \leq 2\binom{k+1}{3}$.

Note that $|\mathcal{C}_k| = \binom{\alpha n}{k}\binom{(1-\alpha)n}{k}$. Let λ be the proportion of sets in $\mathcal{C}_k$ which are good. Let γ_k be defined by

$$\gamma_k = \frac{\mathrm{ex}_B(2k, \mathcal{K}_4^-)}{2\binom{k+1}{3}}.$$

Lemma 10 *With the above notation we have*

$$\pi_2' \leq \frac{(k+1)(1-\epsilon^2)^2(1+\lambda(\gamma_k - 1))}{4k(1-\epsilon\delta)} + O(n^{-1}).$$

Proof: We simply count the number of edges in sets from $\mathcal{C}_k$, yielding

$$\beta m \binom{\alpha n - 2}{k-2}\binom{(1-\alpha)n - 1}{k-1} + (1-\beta)m\binom{\alpha n - 1}{k-1}\binom{(1-\alpha)n-2}{k-2}$$
$$\leq 2\binom{k+1}{3}((1-\lambda) + \gamma_k\lambda)\binom{\alpha n}{k}\binom{(1-\alpha)n}{k}.$$

Using $m = \pi_2'\binom{n}{3}$, $\alpha = (1+\epsilon)/2$, $\beta = (1+\delta)/2$ and rearranging gives the desired inequality. □

The next lemma will allow us to estimate q_1 from our knowledge of the number of good sets in $\mathcal{C}_k$.

Lemma 11 *Let $\mathcal{G}$ be a $\mathcal{K}_4^-$-free 3-graph with vertex set V. For $A \subseteq V$ we define*

$$g(A) = \#\{C \in A^{(4)} \mid \text{ and } C \text{ is good}\}.$$

If $A \subseteq V$ and $g(A) > 0$ then $g(A) \geq |A| - 3$.

Proof: We use induction on $|A| = k$. If $k \leq 4$ the result holds trivially. The result also holds for $k = 5$ (we simply check that any $\mathcal{K}_4^-$-free 3-graph on 5 vertices containing at least one good 4-set in fact contains at least two good 4-sets). So suppose the result holds for $k-1$ and let $A \in V^{(k)}$, $k \geq 6$ and $g(A) > 0$.

Since $g(A) > 0$ there is at least one set $B \in A^{(k-1)}$ such that $g(B) > 0$ and hence our inductive hypothesis implies that $g(A) \geq g(B) \geq |B| - 3 = k - 4 \geq 2$. Counting good 4-sets in $(k-1)$-subsets of A we have

$$\sum_{B \in A^{(k-1)}} g(B) = g(A)(k-4). \tag{2}$$

If we show that $g(B) = 0$ for at most three distinct sets $B \in A^{(k-1)}$ then our inductive hypothesis implies that the LHS of (2) is at least $(k-3)(k-4)$ and so $g(A) \geq k-3$ as required. So we need to show that $g(B) = 0$ for at most three distinct sets $B \in A^{(k-1)}$.

If $B \in A^{(k-1)}$ satisfies $g(B) = 0$ then every good 4-set in A must contain $A \backslash B$. Thus if B_1, B_2, B_3, B_4 are distinct sets in $A^{(k-1)}$, each satisfying $g(B_i) = 0$, then setting $A \backslash B_i = \{a_i\}$ we know that every good 4-set in A contains $\{a_1, a_2, a_3, a_4\}$ and hence $g(A) \leq 1$. But this is impossible since $g(A) \geq 2$. The result then follows by induction on k. $\square$

Our next lemma gives the desired lower bound on q_1 in terms of λ, ϵ and k.

Lemma 12 *If $q_1 = \#\{D \in V^{(4)} : e(D) = 1\}$ and λ, ϵ, k are as above then*

$$q_1 \geq \begin{cases} \frac{\lambda(2k-3)(1-\epsilon^2)^2 n^4}{16k^2(k-1)^2} + O(n^3), & 0 \leq \epsilon \leq \frac{1}{2k-3}, \\ \frac{\lambda(2k-3)(1-\epsilon^2)(1-\epsilon)^2 n^4}{16k^2(k-1)(k-2)} + O(n^3), & \frac{1}{2k-3} \leq \epsilon \leq 1. \end{cases}$$

Proof: Recall that the number of good sets in $\mathcal{C}_k$ is $\lambda|\mathcal{C}_k|$, moreover each such good set contains (by Lemma 11) at least $2k-3$ good 4-sets. Counting good 4-sets in members of $\mathcal{C}_k$ we have

$$\begin{aligned} &(2k-3)\lambda \binom{\alpha n}{k} \binom{(1-\alpha)n}{k} \\ &\qquad \leq q_1 \max\left\{ \binom{\alpha n - 2}{k-2} \binom{(1-\alpha)n - 2}{k-2}, \binom{\alpha n - 1}{k-1} \binom{(1-\alpha)n - 3}{k-3} \right\}. \end{aligned}$$

The bound then follows by rearranging. □

We can now complete the proof of Theorem 1 by showing that $\pi_2' \leq 0.291$.

First note that if $\epsilon \geq 1/4$ then Lemma 6 implies that $\pi_2' \leq 0.28803$. Hence we may assume that $0 \leq \epsilon < 1/4$.

Let $k = 8$, so by Lemma 9 we have $\gamma_k = 173/168$. Now Lemmas 5, 10 and 12 imply that

$$\pi_2' \leq \min\left\{\frac{3(1-\epsilon^2)^2(168+5\lambda)}{1792(1-\epsilon\delta)}, \frac{3\zeta+\sqrt{9\zeta^2-12\zeta\nu}}{2}\right\}, \tag{3}$$

where

$$\zeta = \frac{(1-\epsilon^2)^2}{10-6\epsilon^2-8\epsilon\delta+2\delta+2\epsilon^2\delta^2}$$

and

$$\nu = \begin{cases} \frac{39\lambda(1-\epsilon^2)^2}{25088}, & 0 \leq \epsilon \leq \frac{1}{13}, \\ \frac{39\lambda(1-\epsilon^2)(1-\epsilon)^2}{21504}, & \frac{1}{13} \leq \epsilon \leq 1. \end{cases}$$

Let

$$B = \{(\epsilon, \delta, \lambda) \in \mathbb{R}^3 : 0 \leq \epsilon \leq 1/4, -1 \leq \delta \leq 1, 0 \leq \lambda \leq 1\}.$$

We must now give an upper bound for the maximum of (3) over B. We do this numerically by first computing the value of (3) at all 4×10^{12} points in the regular 3-dimensional lattice with side length 0.00005 in B. This yields the maximum 0.290433. A routine argument bounding the partial derivatives of (3) then implies that $\pi_2' \leq 0.291$ as required.

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