Uncountable families of vertex-transitive graphs of finite degree

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Recently the following question was relayed [1] to the second author: What is the cardinality of the set of vertex transitive graphs of finite degree? Our aim in this short note is to show that there are $2^{\aleph_0}$ such graphs. Our proof is constructive and is based on ideas of B. Neumann [3].

In order to construct a large such set it is natural to turn to Cayley graphs of finitely generated groups (see e.g [2] for definitions and general facts about Cayley graphs). In 1937 B. Neumann [3] proved that the set of finitely generated groups has cardinality $2^{\aleph_0}$. However, since Cayley graphs of non-isomorphic groups can be isomorphic, this alone does not prove that there are the same number of non-isomorphic Cayley graphs.

We will give two uncountable families of groups and generators for which one can prove that the corresponding Cayley graphs are non-isomorphic. Our first family of graphs is based on a variation of the construction of Neumann, and its members are 4-regular. Our second family consists of cubic graphs, for which the isomorphism problem requires a bit more work. From these examples uncountable families of transitive regular graphs of any degree $d \geq 3$ can be constructed as suitable products of our examples with complete, or complete bipartite, graphs.

Example 1. Let $N = (n_1, n_2 \ldots)$ denote a sequence of numbers and let $\{\sigma_{i,j} | i = 1, 2 \ldots; j = 1, \ldots, n_i\}$ be a set of symbols. Now define a permutation group G_N, acting of the set of symbols, generated by the two permutations:

$$\begin{aligned} a &= (\sigma_{11}, \sigma_{12} \ldots \sigma_{1n_1})(\sigma_{21} \ldots \sigma_{2n_2}) \ldots \\ b &= (\sigma_{11}\sigma_{12}\sigma_{13})(\sigma_{21}\sigma_{22}\sigma_{23}) \ldots \end{aligned}$$

Let a sequence M be defined by $m_1 = 8, m_2 = 18, m_k = 2 + \prod_{i=1}^{k-1}(m_i/2)$. We claim that if N is chosen to be a subsequence of M and $G = Cay(N, \{a, b\})$ is the corresponding (undirected, unlabeled) Cayley graph then distinct choices of N gives us non-isomorphic Cayley graphs.

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In order to prove this let us first note that for $t \in M$, $w_t = a^{-(t-2)}ba^{t-2}$ and b commute if and only if $t \notin N$. So if we could identify which edges in G correspond to a and b respectively, and their orientations, we could identify the members of N by checking whether the walks $w_t b$ and bw_t end at the same vertex.

Now, by our choice of M, the only triangles in G correspond to three consecutive b-edges, and so we can partition the edge-set of G into triangles of b-edges and infinite paths of a-edges. The orientation is harder to reconstruct, but we do not actually need to know the orientation in order to identify whether t is in N or not. Indeed, Given that we know the labels of the edges, there are 16 different paths in G which can correspond to $w_t b$, one for each choice of signs in the exponents, and likewise for bw_t. If $t \in N$ then all these paths have distinct endpoints, and if $t \notin N$ some of them will have the same endpoints. We can thus distinguish between the two alternatives of $t \in N$ and $t \notin N$.

So we have found an uncountable family of non-isomorphic 4-regular vertex-transitive graphs. From a result of [3] the underlying groups are also distinct, although this was not used in our proof.

Example 2. Let us define $c = (\sigma_{11}\sigma_{12})(\sigma_{21}\sigma_{22})\dots$ and let us consider the family $G = Cay(N, \{a, c\})$ for N chosen as in Example 1.

As before we want to determine the labels and orientations of the edges of G; then we can, as in the first example, identify the members of N. We observe[1] that the only two kinds of cycles of length 12 in G are those given by $(a^{-2}ca^2c)^2$ or $(a^{-1}cac)^3$. This lets us identify the c-edges as those edges e for which each edge adjacent with e at a given vertex is part of a 12-cycle containing e. This let us partition the edges into a 1-factor of c-edges and infinite paths of a-edges. The orientation of the a-paths can be inferred in the same way as earlier by using that c and a^2ca^{-2} commute.

We mention that this family consists of bipartite graphs, since a has infinite order and the sign of a walk with the same number of a and a^{-1} is given by the number of c's, so any cycle through the identity element must have even length. These graphs also have a 2-factor given by deleting the a-edges, each component of which is an infinite path, and are 3-edge colourable. In this case we do not know whether or not the underlying groups are distinct.

References

[1] Olle Häggström, Personal communication.

[2] Josef Lauri and Raffaele Scapellato, *Topics in graph automorphisms and reconstruction*, Cambridge University Press, Cambridge, 2003.

[3] Bernhard Neumann, Some remarks on infinite groups, *J. London math. Soc.* **12** (1937), 120–127.

[1] Exhausting handwork or simple computer check