ON STABLE CYCLES AND CYCLE DOUBLE COVERS OF GRAPHS WITH LARGE CIRCUMFERENCE

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Abstract. A cycle C in a graph is called *stable* if there exist no other cycle D in the same graph such that $V(C) \subseteq V(D)$. In this paper we study stable cycles in snarks and based on our findings we are able to show that if a cubic graph G has a cycle of length at least $|V(G)| - 9$ then it has a cycle double cover. We also give a construction for an infinite snark family with stable cycles of constant length and answer a question by Kochol by giving examples of cyclically 5-edge connected cubic graphs with stable cycles.

1. Introduction

In this paper a *cycle* is a connected 2-regular subgraph. A *cycle double cover* (usually abbreviated CDC) is a multiset of cycles covering the edges of a graph such that each edge lies in exactly two cycles. The following is a famous open conjecture in graph theory.

Conjecture 1.1 (CDCC)**.** *Every bridgeless graph has a cycle double cover.*

The conjecture is often attributed to Seymour and Szekeres but is probably older. For a survey see [11] or [16]. The conjecture has been proved for some large classes of graphs such as planar graphs and 4-edge-connected graphs. It is also well known that if the conjecture holds for cubic graphs, then it is true for all graphs. Furthermore there are a number of properties a counterexample G to Conjecture 1.1 with the smallest number of edges must have if it exists:

- It is a snark (simple, cubic, $\chi'(G) = 4$, girth at least 5 and cyclically 4-edge connected) ([11]).
- It has girth at least 12 ([8]).
- It has oddness at least 6 ([7]), where *oddness* is the minimum number of odd cycles in any 2-factor.
- It has no hamiltonian path ([15, 9]).

There are various stronger versions of the CDCC. The one we are primarily concerned with in this paper is the following by Goddyn [6].

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Conjecture 1.2 (Strong Cycle Double Cover Conjecture (SCDCC)). *Let G be a bridgeless graph. Then for every cycle C in G there is a CDC that contains C.*

It is easy to see that this holds for 3-edge-colourable cubic graphs, where we in fact even can extend any 2-regular subgraph to a CDC (here it is convenient to consider a 2-regular subgraph as a set of disjoint cycles). Let H be a 2-regular subgraph of a 3-edge-colourable graph G. If $\{M_1, M_2, M_3\}$ are the perfect matchings in a 3-edge-colouring of G, then let $C_i = G - M_i$ for $i = 1, 2, 3$. Now $\{H\Delta C_1, H\Delta C_2, H\Delta C_3, H\}$ is a CDC. Using standard reductions it is also quite easy to see that a minimal counterexample to the SCDCC must be a snark (see e.g. [13] for details).

There are not many families of snarks where the SCDCC is known to be true. One example of a family for which the conjecture is true is the hypohamiltonian snarks for which Fleischner and Häggkvist have shown the following theorem.

Theorem 1.3. [5] *Let $\mathcal{C}$ be a set of disjoint cycles in a hypohamiltonian graph G such that $\cup_{C\in\mathcal{C}} V(C) \neq V(G)$. Then G has a CDC $\mathcal{D}$ such that $\mathcal{C} \subset \mathcal{D}$.*

In the same paper they also show that any cycle that misses at most one vertex can be part of a CDC.

Theorem 1.4. [5] *Let G be a cubic graph on n vertices and let C be a cycle in G. If $|V(C)| \geq n - 1$ then C can be extended to a CDC.*

Using the program *minibaum* by Brinkmann ([1]) we have generated all snarks on at most 32 vertices. In an upcoming paper [2] an improved snark generator is described and the snarks on 34 vertices are studied.

2. Extensions and stable cycles

Let G be a bridgeless cubic graph with a cycle C. We say that C has an *extension* if there exists a cycle C' distinct from C such that $V(C) \subset V(C')$. If a cycle has no extension then it is called a *stable cycle*. A graph that contains stable cycles is called a *stable graph*. Extensions and stable cycles were studied in e.g. [3] and [13] and the following observation show the connection with the SCDCC.

Proposition 2.1. *Let (G, C) be a minimal counter example to SCDCC. Then G is a snark and C is a stable cycle.*

Proof. We have already seen that G must be a snark and therefore it must be 3-connected. Assume that C has an extension D and consider $H = G - (E(C) \setminus E(D))$. H is now a 2-connected graph which contains D. This implies that there is a cycle cover of H that cover $E(D)$ once and all other edges twice. By adding the cycles from $C\Delta D$ we get a cycle cover of G that cover the edges of C once and the rest twice. □

Order	Total number of snarks	Stable snarks	Length of shortest stable cycle
10	1	0	-
18	2	0	-
20	6	0	-
22	20	2	20
24	38	1	22
26	280	7	23
28	2900	25	24
30	28 399	228	26
32	293 059	1456	26

Table 1. Number of small stable snarks compared to the total number of snarks and the length of the shortest stable cycle among the stable snarks of a given order.

Fleischner gave the first construction of a cubic graph with stable cycles [4]. However, his construction does not give rise to snarks since all resulting graphs have cyclic 3-edge cuts. The smallest cubic 3-edge-colourable non-hamiltonian graph (the graph can be found in [14]) is another example of a stable graph, which also happens to be cyclically 4-edge connected, but is of course not a snark. The first example of a snark with stable cycles was given by Kochol in [13] where he gives a construction that yields an infinite family of stable snarks. The smallest snark his construction can produce has 34 vertices and Kochol notes that it has in fact 2 stable cycles. Using a computer search we found another 50 stable cycles in the same graph. In the same paper Kochol gives the following problem: *Construct cyclically 5- or 6-edge connected snarks with stable cycles.* Using a computer search we were able to show the following.

Observation 2.2. *There are four stable cyclically 5-edge connected snarks on* 32 *vertices and there are no such snarks on fewer vertices.*

The four snarks are shown in figure 1 and can also be found in the appendix.

By computer search we tested all snarks on less than 34 vertices for the presence of stable cycles. The number of stable snarks of a given order can be found in Table 1. We have also verified that each of the stable cycles is part of some CDC.

Observation 2.3. *Every stable cycle in the snarks of order at most 32 can be included in some CDC.*

Corollary 2.4. *The strong cycle double cover conjecture holds for all graphs of order at most* 32.

Using the fact that the SCDCC holds for all graphs on less than 34 vertices it is easy to show that if a graph has a cycle of length $n-9$ or longer, where n is the number of vertices, then it must have a CDC.

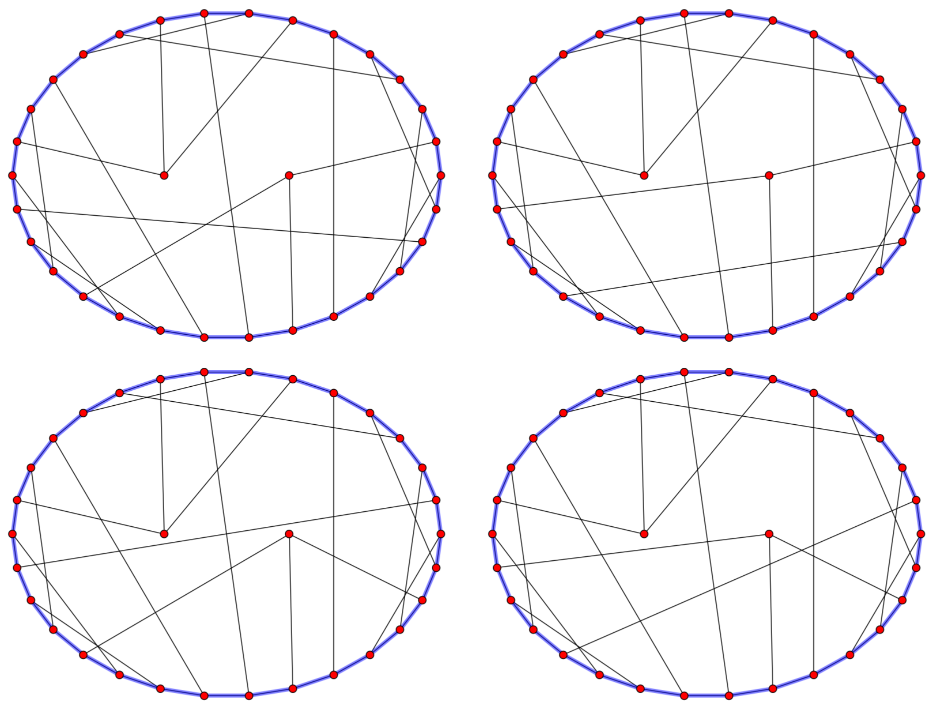

FIGURE 1. The four cyclically 5-edge connected snarks on 32 vertices with stable cycles. The outer cycle is the stable one. Note that all graphs are the same except for the fact that the endpoints of two edges changes places. The graphs can also be found in the appendix where they are represented in Brendan McKay's graph6-format.

Theorem 2.5. *Let G be a bridgeless cubic graph of order n. If G has a cycle of length at least $n-9$ then G has a CDC.*

Proof. Assume that G has a cycle C such that $|C| > n-10$. If there exists a vertex $v \in V(G) \setminus V(C)$ such that $N_G(v) \subset V(C)$, then we let H be the subgraph of G induced by the vertices of C and v. It follows from theorem 1.4 that there is a CDC $\mathcal{D}$ in H such that $\mathcal{D} \ni C$. Now consider the graph G' where we remove all the chords of C and the vertex v with incident edges and then contract one of the edges incident to each of the 2-valent vertices on the cycle (the endpoints of the removed chords and the edges incident to v). G' is now a cubic graph with an induced cycle C' and $|C'| = |\partial(V(G) \setminus (V(C) \cup \{v\}))| \leq 24$ since G is cubic and $|V(G) \setminus (V(C) \cup \{v\})| \leq 8$.

If no such vertex v exists we have that $|\partial(V(G) \setminus V(C))| < 16$ and we can choose $H = G[C]$ which is of course hamiltonian and therefore has a CDC $\mathcal{D} \ni C$ and we construct G' as before by removing all the chords in C and contracting one edge incident to each of the 2-valent vertices.

Since $|V(G')| \leq 32$, using corollary 2.4 we get a CDC $\mathcal{D}'$ of G' where $\mathcal{D}' \ni C'$. Subdivide the edges in G' to get G minus the chords of C and let $\mathcal{D}''$ be the corresponding CDC. Now $(\mathcal{D} \cup \mathcal{D}'') \setminus C$ is the desired CDC of G. □

What we in fact show in the proof is something slightly stronger than the statement of the theorem: if $|\partial_G(V(G) \setminus V(C))| + |V(G) \setminus V(C)| \leq 32$ then G has a CDC and if there is a vertex with all neighbours on the cycle then $|\partial_G(V(G) \setminus V(C))| + |V(G) \setminus V(C)| \leq 36$ is a sufficient condition for a CDC.

In [8] Huck shows that the girth of the smallest counterexample to CDCC is at least 12. His proof is heavily based on computers for checking reducible configurations. There are constructions for snarks with high girth [12]. It is worth noticing that all known constructions for snarks with high girth give quite large graphs. It might be possible to improve Huck's result using more computational power, but it would require some effort.

The authors are not aware of any snark construction that can simultaneously give high girth and low circumference and therefore pose the following problem.

Problem 1. Give a construction that produces snarks of girth at least g and circumference at most $n - g$ where g is a constant.

3. Short stable cycles

As we can see in Table 1, there are quite short stable cycles in some snarks. By using Isaac's dot product construction [10] on a snark with a non-dominating stable cycle (see fig 2), we were able to show that in fact there is an infinite family of snarks with stable cycles of length 24.

Theorem 3.1 (Isaacs [10])**.** *Let G_1 and G_2 be two snarks with edges $e_1 = x_1x_2, e_2 = x_3x_4 \in E(G_1)$, $f = v_1v_2 \in E(G_2)$ and $N_{G_2-f}(v_1) = \{u_1, u_2\}$, $N_{G_2-f}(v_2) = \{u_3, u_4\}$. Then the cubic graph obtained by deleting e_1 and e_2 from G_1 and the vertices v_1 and v_2 from G_2 and then adding the edges $\{x_1u_1, x_2u_2, x_3u_3, x_4u_4\}$ is a snark.*

Let G be the graph depicted in figure 2 and C be the stable non-dominating cycle of length 24. Remove the edge v_1v_2 (the dotted edge) and its end vertices and let G' be the resulting graph. We now have 4 vertices of degree 2: $\{u_1, u_2, u_3, u_4\}$. Adding edges u_1u_2 and u_3u_4 cannot give rise to any new extensions of C since that would also correspond to an extension in G. If we instead add edges $\{u_1u_4, u_2u_3\}$ or $\{u_1u_3, u_2u_4\}$ it is not as obvious that we do not get any any extensions of C but by careful case analysis either by hand or by computer it is possible to see that this is indeed the case.

Take a copy of the Petersen graph (or some other snark for that matter) and remove two independent edges x_1x_2 and x_3x_4. Now x_1, x_2, x_3 and x_4 are vertices of degree 2. Connect the two graphs by adding the edges $D = \{u_1x_1, u_2x_2, u_3x_3, x_4u_4\}$. We call the resulting graph H. By theorem 3.1 we know that H is a snark. Since adding edges in some way between the

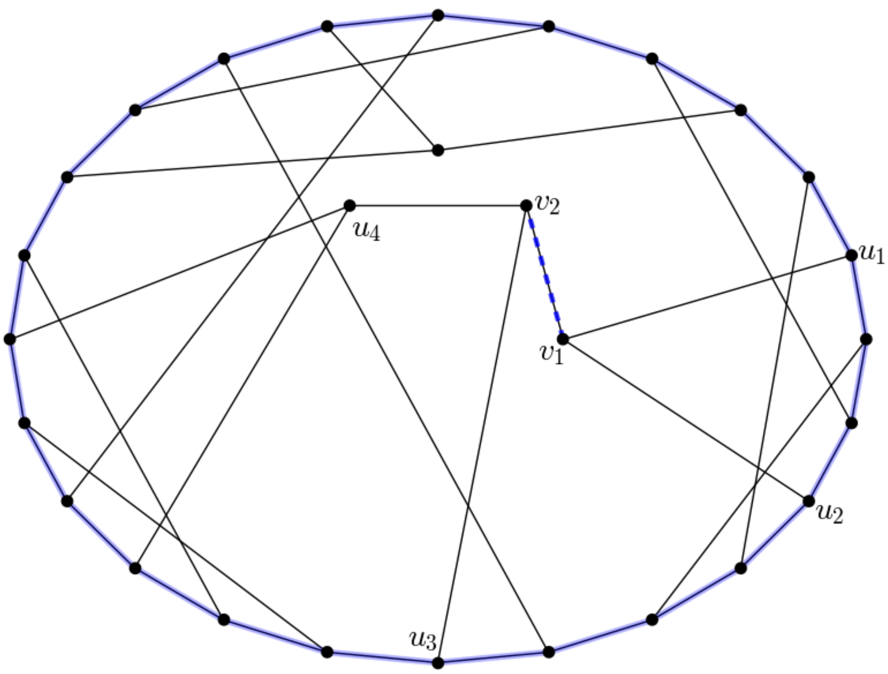

FIGURE 2. A snark on 28 vertices with a stable cycle of length 24 that is not dominating.

vertices of degree 2 in G' to form a cubic graph did not give rise to any extension of C, we cannot get any extension over the edge cut D either. Hence C must also be a stable cycle in H. By iteratively using theorem 3.1 on some edges that do not lie on the same side of the edge cut D as C does, we can construct even bigger snarks where C is still a stable cycle. We get the following theorem:

Theorem 3.2. *There is an infinite family of snarks with stable cycles of length 24.*

Note that this construction does not work for all non-dominating stable cycles and all edges with no endpoint on the cycle. If we start with the edge v_2u_4 instead, deleting this edge and its endpoints and then adding the appropriate edges to get a new cubic graph we get in fact an extension of C.

4. SEMIEXTENSIONS

It was shown by Esteva and Jensen in [3] that a minimum counter example to SCDCC must also satisfy a stronger condition than having stable cycles. Let C be a cycle in a graph G. We say that C has a *semiextension* if there exists another cycle $D \neq C$ in G such that for every path $P = xv_1v_2 \dots v_ky$ where $x \in V(C) \setminus V(D)$, $y \in V(C) \cup V(D)$ and $v_1, \dots, v_k \in V(G) \setminus (V(C) \cup$

$V(D)$) we have another $x-y$-path P' where $E(P') \subset E(C)\Delta E(D)$. A cycle with no semiextension is called a *superstable cycle*.

Theorem 4.1 (Esteva and Jensen [3]). *Let (G, C) be a minimal counterexample to SCDCC. Then C must be a superstable cycle.*

They also give the following conjecture.

Conjecture 4.2. *There are no 2-connected cubic graphs with superstable cycles.*

Using a computer we have verified this conjecture for all snarks of order at most 32.

Observation 4.3. *No snark on 32 vertices or less has superstable cycles.*

We also note that the short stable cycle in the graphs from Theorem 3.2 will always have semiextensions.

5. Acknowledgment

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References

1. Gunnar Brinkmann, *Fast generation of cubic graphs*, J. Graph Theory **23** (1996), no. 2, 139–149. MR MR1408342 (97e:05179)
2. Gunnar Brinkmann, Jan Goedgebeur, Jonas Hägglund, and Klas Markström, *Generation and properties of snarks*, In preparation.
3. Enrique García Moreno Esteva and Tommy R. Jensen, *On semiextensions and circuit double covers*, J. Combin. Theory Ser. B **97** (2007), no. 3, 474–482. MR MR2305899 (2008f:05150)
4. Herbert Fleischner, *Uniqueness of maximal dominating cycles in* 3*-regular graphs and of Hamiltonian cycles in* 4*-regular graphs*, J. Graph Theory **18** (1994), no. 5, 449–459. MR MR1283310 (95b:05121)
5. Herbert Fleischner and Roland Häggkvist, *Circuit double covers in special types of cubic graphs*, Discrete Math. **309** (2009), no. 18, 5724–5728. MR 2567976
6. Luis Armando Goddyn, *Cycle covers of graphs*, Ph.D. thesis, University of Waterloo, 1988.
7. Roland Häggkvist and Sean McGuinness, *Double covers of cubic graphs with oddness 4*, J. Combin. Theory Ser. B **93** (2005), no. 2, 251–277. MR MR2117938 (2005m:05175)
8. Andreas Huck, *Reducible configurations for the cycle double cover conjecture*, Proceedings of the 5th Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1997), vol. 99, 2000, pp. 71–90. MR MR1743825 (2000m:05185)
9. Andreas Huck and Martin Kochol, *Five cycle double covers of some cubic graphs*, J. Combin. Theory Ser. B **64** (1995), no. 1, 119–125. MR MR1328296 (96b:05133)
10. Rufus Isaacs, *Infinite families of nontrivial trivalent graphs which are not Tait colorable*, Amer. Math. Monthly **82** (1975), 221–239. MR 0382052 (52 #2940)
11. François Jaeger, *A survey of the cycle double cover conjecture*, Cycles in graphs (Burnaby, B.C., 1982), North-Holland Math. Stud., vol. 115, North-Holland, Amsterdam, 1985, pp. 1–12. MR MR821502 (87b:05082)

12. Martin Kochol, *Snarks without small cycles*, J. Combin. Theory Ser. B **67** (1996), no. 1, 34–47. MR MR1385382 (97d:05114)
13. ______, *Stable dominating circuits in snarks*, Discrete Math. **233** (2001), no. 1-3, 247–256, Graph theory (Prague, 1998). MR MR1825619 (2001m:05193)
14. Klas Markström, *Extremal graphs for some problems on cycles in graphs*, Proceedings of the Thirty-Fifth Southeastern International Conference on Combinatorics, Graph Theory and Computing, vol. 171, 2004, pp. 179–192. MR 2122107
15. Michael Tarsi, *Semiduality and the cycle double cover conjecture*, J. Combin. Theory Ser. B **41** (1986), no. 3, 332–340. MR MR864580 (87m:05060)
16. Cun-Quan Zhang, *Integer flows and cycle covers of graphs*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 205, Marcel Dekker Inc., New York, 1997. MR MR1426132 (98a:05002)

Appendix

The four cyclically 5-connected stable snarks on 32 vertices represented in McKay's graph6-format:

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_sP@P?WC?I?a?_?W?A??O?@??A??C??A???c??G_??_??A???AA??A????__??DA???Q???@E????`????AS
_sP@P?WC?I?a?_?W?A??O?@??A??C??A???c??G_??_??A???AA??A????_???DA???Q???@E????h????BO
_sP@P?WC?I?_?`?W?A??_?A??C??C??A???a??G???c??AC??AC??A????__??GA???Oc??@@????o????@K
_sP@P?WC?I?_?`?W?A??_?A??C??C??A???a??G???c??AC??A???A????__??GC???O_??@@O???q????`O
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