The Multivariate Ising Polynomial of a Graph

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Abstract. In this paper we discuss the two variable Ising polynomial in a graph theoretical setting. This polynomial has its origin in physics as the partition function of the Ising model with an external field. We prove some basic properties of the Ising polynomial and demonstrate that it encodes a large amount of combinatorial information about a graph. We also give examples which prove that certain properties, such as the chromatic number, are not determined by the Ising polynomial. Finally we prove that there exists large families of non-isomorphic planar triangulation with identical Ising polynomial.

1. Introduction

In graph theory a number of different polynomials associated to a graph has been introduced over the years, and it has repeatedly turned out that they are in fact specialisations of the flagship of this armada of polynomials, the Tutte polynomial. The Tutte polynomial was introduced by Tutte in 1947 [Tut47] after he had observed that a number of interesting graph parameters satisfied similar recursive identities. The Tutte polynomial contains important polynomials such as the chromatic polynomial [RT88], the Jones polynomial of a knot and the reliability polynomial of a network, see [Wel93] for a survey.

In the 1970's it was also realised that the Tutte polynomial had an important role to play in statistical physics as well. In 1925 Ising and his thesis advisor Lenz [Isi25] introduced the Ising model for magnetism. In this model a "spin" of value ± 1 is assigned to every vertex in a graph G. An edge with equal spins at the endpoints is given an energy of 1 and one with unequal endpoints an energy of -1. The total sum of the spin is called the magnetisation. Summing over all such spin assignments we get a generating function $Z(G, x, y)$, here called the *Ising polynomial*, where the coefficient of $x^i y^j$ counts the assignments with energy i and magnetisation j. The Ising model now studies how magnetisation and energy are correlated under a suitable probability measure on the set of spin assignments.

This model was later extended to the Potts modell, which allows more than just two values of the spin. In 1972 Fortuin and Kasteleyn [FK72] introduced a new representation for the Potts model, where the magnetisation was not included, called the random-cluster model, see [Gri06] for a textbook treatment. It was then realised that the generating function controlling this model is in fact equivalent to the Tutte polynomial. See [Sok05] for a recent survey of the many results connecting the Tutte polynomial and the random cluster model to various topics in graph theory and physics.

For the Ising model we can find the restricted polynomial $Z(G,x,1)$ from the Tutte polynomial of G. However, since this polynomial no longer contains the total magnetisation of the spins the random cluster model does not capture all properties of the original Ising model. There has been many interesting results specifically for the Ising model in the physics literature over the years. Such as [LY52] where it is shown that for a fixed x the zeros of $Z(G,x,y)$ lie on a circle in the complex plane, but the full Ising polynomial $Z(G,x,y)$ has not received the same attention in the graph theoretical literature.

Our current aim is to demonstrate that the Ising polynomial $Z(G,x,y)$ is an interesting polynomial from a graph theoretical perspective and study some of its properties. Interestingly we will show that although $Z(G,x,1)$ is determined by the Tutte polynomial of G the full polynomial $Z(G,x,y)$ is not, e.g. unlike the Tutte polynomial it is not trivial function when G is a tree. In the other direction we will also show that the Tutte polynomial is not determined by the Ising polynomial. We thus have a graph polynomial which, although related to, is essentially different from the Tutte polynomial. Just like for the Tutte polynomial there are known graph polynomials which are contained in the Ising polynomial, e.g. the matching polynomial and for regular graphs the independence polynomial. Perhaps the first two members of a new armada.

2. Definitions and relations

We will now give the formal definition of the Ising polynomial. In fact we will give two equivalent definitions and demonstrate that the Ising Polynomial is also equivalent to a second polynomial related to eulerian subgraphs of G. The first definition is the original physics definition in terms of "spin states" on the vertex set of G and the second a reformulation of this definition in terms of edge cuts in G.

2.1. The state sum definition.

We will first define a few terms needed for the definition of the Ising polynomial, due to the Ising polynomials origin as a physical model for phase transitions in magnetic materials the terminology has a physical flavour. We let G be a simple graph, $V(G)$ it's vertex set, and $E(G)$ its edge set.

A *state* σ on G is a function $\sigma : V(G) \to \{-1, 1\}$, the value of σ at a vertex v is called the *magnetisation* of v.

Definition 2.1. Given a state σ the energy $E(\sigma, e)$ of an edge $e = (u, v)$ in G is $E(\sigma, e) = \sigma(u)\sigma(v)$, and the energy $E(\sigma)$of the state σ is the sum of the energies of the edges, that is

$$E(\sigma) = \sum_{e \in E(G)} E(\sigma, e)$$

Definition 2.2. The magnetisation $M(\sigma)$ of a state σ is the sum of the magnetisations of all the vertices in G, that is,

$$M(\sigma) = \sum_{u \in V(G)} \sigma(u)$$

We let Ω denote the set of all states on G.

We can now define the *Ising polynomial*:

Definition 2.3 (The Ising Polynomial)**.** The Ising polynomial is

$$Z(G, x, y) = \sum_{i,j} a_{i,j} x^i y^j = \sum_{\sigma \in \Omega} x^{E(\sigma)} y^{M(\sigma)}$$

Here we can note two things about $Z(G, x, y)$. First, $Z(G, x, y)$ is a Laurent-polynomial rather than a polynomial, it can have monomials with negative, but integer, powers. Second it is also the generating function for the number of states on G with given magnetisation and energy. The following simple lemma will later be useful.

Lemma 2.4. *The exponents of x in $Z(G, x, y)$ is at most n, and at least $-n$. The exponents of y in $Z(G, x, y)$ is at most m, and at least $-m$.*

Another way to look at the state sum definition is to consider a state on G as a graph homomorphism from G to a weighted K_2 with loops on both vertices, the vertices have weights y and y^{-1}, the loops have weight x and the ordinary edge weight x^{-1}. Recently [FLS07] has shown that a large set of models from statistical mechanics, having a property called reflection positivity, are equivalent to counting weighted graph homomorphisms in this way.

2.2. Cuts and T-joins. A more graph theoretical interpretation of the Ising polynomial can be given in terms of cuts. The values of a state σ defines a bipartition of the graph G such that all the edges with negative energy have one endpoint in each partition. The coefficient $a_{i,j}$ thus enumerates edge cuts in G such that there are $(m-i)/2$ edges in the cut and $(n-j)/2$ vertices in one part of the bipartition.

Definition 2.5 (Cut)**.** A cut $[S, \bar{S}]$, where $\bar{S} = V \setminus S$, in a graph $G = (V, E)$ is a subset of edges, induced by a partition $S \cup \bar{S} = V$ of the vertices of G, that have one endpoint in S and the other in $\bar{S}$. We let $\big|[S, \bar{S}]\big|$ denote the number of edges with exactly one endpoint in S.

In this notation we can give the following equivalent definition of the polynomial

Definition 2.6 (The Ising Polynomial)**.** The Ising polynomial is now defined as

$$Z(G, x, y) = \sum_{i,j} a_{i,j} x^{n-2i} y^{m-2j},$$

where $a_{i,j}$ counts the number of cuts $[A, B]$ of $V(G)$ such that $|A| = j$ and $|[A, B]| = i$.

A polynomial closely related to the Ising polynomial is what we call the *van der Waerden polynomial.* This polynomial is a multivariate generalisation of a polynomial studied by van der Waerden in [vdW41].

Definition 2.7 (van der Waerden Polynomial)**.**

$$W(G, t, u) = \sum_{i,j} b_{i,j} u^i t^j,$$

where $b_{i,j}$ is the number of subgraphs of G with i edges and j vertices of odd degree.

As we will now prove, the two polynomials are in fact equivalent and can be transformed into each other by a nontrivial change of variables. This transformation is best formulated using T-joins.

Definition 2.8 (T-join)**.** A T-join (T, A) in a graph $G = (V, E)$ is a subset $T \subseteq V$ of vertices and a subset $A \subseteq E$ of edges such that the vertices in T are incident with an odd number of edges in A and the vertices in $V \setminus T$ are incident with an even number of edges from A.

The relation between the number of cuts and the number of T-joins can now be given.

Theorem 2.9. *Let $G = G(V, E)$ be a graph, $a_{i,j}$ the number of cuts with i vertices in one part and j edges between the parts and let $b_{i,j}$ be the number of T-joins with i odd vertices and j edges. Then*

$$\sum_{ij} b_{i,j} x^i y^j = 2^{-|V|} \sum_{ij} a_{i,j} (1-x)^i (1+x)^{|V|-i} (1-y)^j (1+y)^{|E|-j}$$

Proof. Fix a subset $T \subseteq V$ of vertices and a subset $A \subseteq E$ of edges from the graph $G = G(V, E)$. Let $S \subseteq V$ be another subset of vertices and $[S, \bar{S}]$ be the cut defined by the edges from S to $\bar{S} = V \setminus S$. Let the weight of the vertices in $T \cap S$ be $-x$, the weight of the vertices in $T \cap \bar{S}$ be x, the weight of the edges from A that lies in the cut $[S, \bar{S}]$ be $-y$ and the rest of the edges from A have weight y. Let the total weight of (T, A) with respect to the cut $[S, \bar{S}]$ be the product of the weights of the edges and vertices in (T, A). We say that the weight is positive if the coefficient of $x^{|T|} y^{|A|}$ is positive and negative otherwise.

If (T, A) is not a T-join, there does exist a vertex v that either is incident with an odd number of edges from A and does not belong to T, or is incident with an even number of edges from A and belongs to T. In either case we have a bijection between cuts with $v \in S$ and $v \notin S$ (we simply move the vertex v between S and $\bar{S}$) that give the same weight to our choice of (T, A) except that they have different signs and thus cancel out when summed over all cuts.

If (T, A) indeed is a T-join the weight will always be positive since we either have an even number of vertices in $T \cap S$ and an even number of edges crossing the cut or an odd number of vertices in $T \cap S$ and an odd number of edges crossing the cut. All in all we end upp with an even number of minus signs and thus a positive weight.

If we now sum over all choices (T, A) and S we will count each T-join $2^{|V|}$ times. If we rearrange our summation (*i.e.* we first choose a cut and then go through all choices of T and A) we get the theorem. □

The same reasoning also gives an inverse relation.

Corollary 2.10. *With the same notation as in Theorem 2.9 we have*

$$\sum_{ij} a_{i,j} x^i y^j = 2^{-|E|} \sum_{ij} b_{i,j} (1-x)^i (1+x)^{|V|-i} (1-y)^j (1+y)^{|E|-j}$$

Proof. If we choose a T-join instead of a cut the weight of (T, A) will always be positive if and only if (T, A) is a cut. In other cases the contribution will once again cancel out. □

3. Basic properties of the Ising polynomial

For graphs with several components the Ising polynomial factors in terms of the polynomials of the components.

Theorem 3.1. *If G has components G_1 and G_2 then $Z(G) = Z(G_1)Z(G_2)$, and $W(G,t,u) = W(G_1,t,u)W(G_2,t,u)$*

Proof. Immediate by Definition 2.6 since given any pair of bipartitions (A_1, B_1) and (A_2, B_2) of G_1 and G_2 respectively we get a cut $[A, B] = [A_1 \cup A_2, B_1 \cup B_2]$ with $|[A, B]| = |[A_1, B_1]| + |[A_2, B_2]|$. □

Recall that the *join* of two graph G_1 and G_2 is the graph obtained by taking the disjoint union of the two graphs and adding an edge from every vertex in G_1 to every vertex in G_2.

Theorem 3.2. *If the a graph G is the join of two graphs G_1, and G_2 with*

$$Z(G_1, x, y) = \sum_{i,j} a^1_{i,j} x^i y^j$$

and

$$Z(G_2, x, y) = \sum_{i,j} a^2_{i,j} x^i y^j$$

then

$$Z(G) = \sum_{i,j;k,l} a^1_{i,j} a^2_{k,l} x^{i+k+jl} y^{j+l}$$

Proof. Let σ_1 be a state on G_1 with energy i and magnetisation j, and σ_2 a state on G_2 with energy k and magnetisation l. For every such pair of states there is a state σ on G whose restriction to the subgraph G_1 is σ_1, and likewise for G_2 and σ_2.

The state σ has magnetisation $j + l$, and we also want to determine its energy. We can assume that the state σ_1 has a vertices with spin -1 and b, with spin $+1$, and that the state σ_2 has c vertices with spin -1 and d, with spin $+1$. This means that among the edges with exactly one endpoint in G_1 there will be $ac + bd$ edges with positive energy and $ad + bc$ with negative energy. This gives a total contribution to the energy from these edges of

$$ac + bd - ad - bc = (a - b)(c - d) = jl.$$

The edges within the two subgraphs contribute i and k to the energy and we get a total of $i + k + jl$.

Summing over all pairs of states σ_1, σ_2 gives the Ising-polynomial of G as stated in the theorem. □

The Ising polynomial of the complete graph has a particularly simple structure which we will make use of.

Example 3.3. Consider the complete graph K_n. Any state with a negative spins will have magnetisation $n-a$ and energy $\binom{a}{2} + \binom{n-a}{2} - (n-a)a$. Thus the Ising polynomial of a complete graph with n vertices will be

$$Z(K_n, x, y) = \sum_{a=0}^{n} \binom{n}{a} x^{\binom{a}{2}+\binom{n-a}{2}-(n-a)a} y^{n-a}$$

The state sum definition allows us to construct the Ising polynomial of the complement of a graph from the Ising polynomial of the graph and Ising polynomial of K_n.

Theorem 3.4. *Let G be a graph on n vertices. The Ising polynomial of the complement $\overline{G}$ of G is given by*

$$Z(\overline{G}) = \sum_{ij} a_{i,j} x^{\binom{a}{2}+\binom{n-a}{2}-(n-a)a-i} y^j,$$

where $a = \frac{n-j}{2}$.

Proof. Let σ be a state on G with energy i and magnetisation j. Since G and $\overline{G}$ have the same vertex set σ will also be a state on $\overline{G}$ with the same magnetisation.

Given a state on a graph and its complement we see that the energies of the two must sum to the energy of the corresponding state on the complete graph on n vertices, since the two graphs partition of the edges of the complete graph.

The number of negative spins in the state is $a = \frac{n-j}{2}$ and so we find that the energy of the state on the complement of G is

$$\binom{a}{2} + \binom{n-a}{2} - (n-a)a - i$$

as required. □

It is also possible to give recursive expressions for the Ising Polynomial by using partial states. We say that (σ, H) is a *partial state* on G if H is a subgraph of G and σ is a state on H. Given a state σ on G and a subgraph $H \subset G$ we let $\sigma|_H$ denote the restriction of σ to $V(H)$.

Definition 3.5.

$$Z(G,x,y,(\sigma',H)) = \sum_{\sigma\in\Omega,\sigma|_H=\sigma'} x^{E(\sigma)}y^{M(\sigma)}$$

Given a set $H \subset V(G)$ let the components of $G \setminus H$ be $G_1,\dots,G_k$ and let G_i' be the subgraph induced by $V(G_i)\cup V(H)$. With this we can now express the Ising polynomial of G as

$$Z(G,x,y) = \sum_{\sigma'\in\Omega(H)} \frac{\prod_{i=1}^k Z(G_i',x,y,(\sigma',H))}{Z(H,x,y,(\sigma',H))^{k-1}}$$

Note that $Z(H,x,y,(\sigma',H))$ is simply a monomial. This relation is very similar to that which exist for the chromatic polynomial when H is a clique cut-set, see [RT88].

This relation becomes especially useful if $H=v$ is simply a cut-vertex in G. For this case we find

$$Z(G,x,y) = \sum_{\sigma'(v)\in\Omega(v)} \frac{\prod_{i=1}^k Z(G_i',x,y,(\sigma',H))}{Z(H,x,y,(\sigma',H))^{k-1}},$$

where $Z(H,x,y,(\sigma',H)$ takes only the values y and y^{-1}. Further we can use the fact that

$$Z(H,x,y,(-\sigma',H)) = Z(H,x,y^{-1},(\sigma',H))$$

to halve the number of distinct polynomials we need to compute. It is easy to check that this gives a polynomial time algorithm for computing the Ising polynomial of a tree. In fact, using the general dynamic programming methods described in [Ree03] it is straightforward to prove that

Theorem 3.6. *For every fixed k there exist a polynomial time algorithm for computing the Ising polynomial of a graph with treewidth at most k.*

4. Graph Invariants

The Ising polynomial of a graph G encodes a large amount of information about G. Our next aim to to give some examples of useful graph properties which can be found from the coefficients of either $Z(G,x,y)$ or $W(G,t,u)$. We first have two theorems which apply to general graphs.

Theorem 4.1. *The following properties can be deduced directly from the ising polynomial $Z(G,x,y)$,*

(1) *The degree-sequence of G.*

(2) *The number of components of G and their size.*
(3) *The smallest edge-connectivity of the components of G.*
(4) *The size of a maximal edge-cut in G.*
(5) *Whether G is bipartite or not.*

Proof. (1) Given a vertex v consider the state where $\sigma(v) = -1$, and $\sigma(y) = 1$ for other vertices y. This state has magnetisation $n - 2$ and energy $m - 2d(v)$. These are the only states with magnetisation $n - 2$ and thus the collection of monomials with energy $n - 2$ is the generating function for the degree sequence of G.

(2) Let p(y) be the polynomial consisting of the monomials of $Z(G, x, y)$ with energy m, divided by x^m. Assume that G has components $G_1, \ldots, G_t$ and $n_i = |V(G_i)|$. The polynomial $p(y)$ is of the form

$$p(y) = \prod_{i=1}^{t} (y^{n_i} + y^{-n_i}).$$

Since $(y^{n_i} + y^{-n_i})$ has a unique factorisation we can find the numbers n_i simply by factoring $p(y)$, as a polynomial over the integers.

(3) Let $[S, \bar{S}]$ be a minimal edge cut in G and let σ be the state which is 1 on S and -1 on $\bar{S}$. This state will have energy $m - 2\left|[S, \bar{S}]\right|$, and since it is a minimal cut all other states will an energy m or less than $m - 2\left|[S, \bar{S}]\right|$. Thus we can find the edge-connectivity by looking at the monomial in $Z(G, x, y)$ with the second highest energy.

(4) Let $[S, \bar{S}]$ be a minimal edge cut in G, let a be the number of edges induced by S plus the number of edges induced by $\bar{S}$, and let $b = \left|[S, \bar{S}]\right|$. We now know that $a + b = m$, and that if σ is the state which is 1 on S and -1 on $\bar{S}$ then the energy of σ will be $a - b$. So, given a monomial $x^i y^j$ we know that this corresponds to a cut of size $\frac{m-j}{2}$. By maximising this over all monomials in $Z(G, x, y)$ we can find the size of a maximum cut in G.

(5) G is bipartite if and only if the largest size of an edge cut is m □

Theorem 4.2. *The following properties can be deduced directly from the van der Waerden polynomial $W(G, t, u)$,*

(1) *The girth of G.*
(2) *The matching polynomial of G.*

(3) *The number of subgraphs of G with i edges and no vertices of odd degree. The generating function is given by $W(G, t, 0)$*

Proof. (1) Let j be the smallest number for which there is a monomial of the form $t^i u^0$, $i > 0$, in $W(G, t, u)$. This is a minimal subgraph of even degree, and thus a shortest cycle in G.

(2) A matching in G with k edges is the only subgraph with exactly $2k$ vertices of odd degree and exactly k edges. Thus we can find the matching polynomials by reading the monomial of the form $t^k u^{2k}$ in $W(G, t, u)$.

(3) Trivial

□

For regular graphs we can deduce even more, thanks to the fact that we can make a direct connection between energy and magnetisation for certain types of states on G. Our main example here is the clique number and independence number of G, and in fact the numbers of cliques and independent sets.

Theorem 4.3. *For an r-regular graph the Ising polynomial $Z(G, x, y)$ determines the the number of k-cliques and the number of independent sets of size k.*

Proof. If we can find the number of independent sets of size k then we can find the number of cliques by using Theorem 3.4.

Let X be a set of k vertices and let σ be the state which is -1 on X and 1 on $\bar{X}$. The energy of σ will be $m - 2rk + a$, where a is the number of edges induced by X. Thus, if X is an independent set the energy will be $m - 2rk$, and this is the only type of state with this energy and magnetisation $n - 2k$. Accordingly the number of independent sets of size k will be coefficients of $x^{m-2rk} y^{n-2k}$ in $Z(G, x, y)$. □

The *independence polynomial* of a graph is the polynomial $I(G, x) = \sum_i a_i x^i$ where a_i counts the number of independent sets on i vertices in G. Since the Ising polynomial allows us to count the number of independent sets of a given size this means that we can find $I(G, x)$, given $Z(G, x, y)$ for a regular graph G. The independence polynomial is an important function in both statistical mechanics and general probabilistic combinatorics, but a deeper discussion of its uses is beyond the scope of this paper. However, the interested reader can find an extensive discussion and survey in [SS05, SS06].

For trees both edge-connectivity, girth, 2-factors and even-subgraphs are trivial and can thus not be used to distinguish two trees from each other. It is also an easy exercise to show that $Z(G, x, 1)$ is the same for all trees on n vertices. However due to the special structure of trees a few other properties can be deduced from the Ising polynomial of a tree.

Lemma 4.4. *Given the Ising polynomial of a tree T we can find the following,*

1. *The diameter of T.*
2. *The characteristic polynomial of G.*

Proof. (1) The diameter of T is given by the length of the longest path in T. From the van der Waerden polynomial we can find the number of subgraphs of T with i edges and two vertices of odd degree. Since T has no cycles such a subgraph is a path and the diameter is given by the largest i for which $b_{i,2}$ is non-zero.

(2) By 4.2 we can find the matching polynomial of T and by a standard theorem, see [God93], the characteristic polynomial and the matching polynomial are equal for trees.

□

5. Examples and Results for small graphs

Given that the Ising polynomial, together with the van der Waerden polynomial, determines many non-trivial properties of G it is natural to ask which properties are not captured by $Z(G, x, y)$. Just like for the other common graph polynomials it turns out that the Ising polynomial is not a complete graph invariant, i.e. there are non-isomorphic graphs with the same Ising polynomial. We will say that two such graphs are *isomagnetic*, and otherwise that the graph is magnetically unique.

An exhaustive computer search among the small graphs proves that the smallest graphs which are not magnetically unique have 7 vertices. There are 36 such graphs, and the equivalence classes form 18 graph pairs. One such pair is shown in Figure 1.

By Lemma 4.4 two isomagnetic trees must also be co-spectral, i.e. their adjacency matrices must have the same eigenvalues, but as demonstrated in [McK77] most trees are not determined by their spectrum. For the Tutte polynomial the situation is even simpler, all trees on n vertices have the same Tutte polynomial. In view of these observations it is interesting to see how well the Ising polynomial separates trees. In Table 1 we give

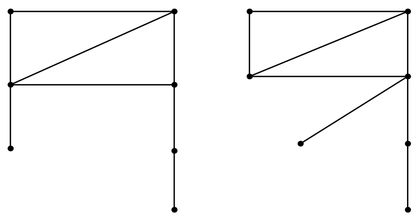

FIGURE 1. The smallest non-unique graphs

the results of a computation for all trees on at most 18 vertices. As we can see the smallest pair of isomagnetic trees, shown in Figure 2, have 11 vertices, which is larger than the smallest examples of co-spectral trees [McK77]. The proportion of trees which are magnetically unique does not seem to fall as n increases, but this could just be because we are working with too small trees. Further, all equivalence classes up to 18 vertices are pairs.

n	$\|T_n\|$	
11	235	2
12	551	4
13	1301	2
14	3159	14
15	7741	8
16	19320	36
17	48629	52
18	123867	92

TABLE 1. T_n denotes the set of non-isomorphic trees on n vertices. The last column gives the number of trees which are not magnetically unique.

Since there are graphs which have the same Tutte polynomial, and also the same characteristic polynomial, but different Ising polynomials it is natural to look for examples of isomagnetic graphs which differ in the first two polynomials. In Figure 3 we show one such pair. These two graphs have the same Ising polynomial, in 5.1, we give the half of the polynomial with non-negative y-exponents.

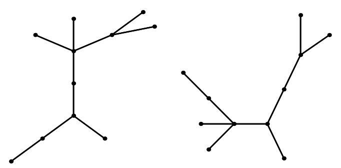

FIGURE 2. Isomagnetic trees on 11 vertices

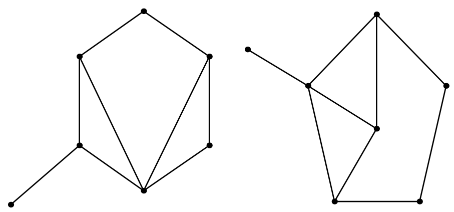

FIGURE 3. Isomagnetic graphs with different Tutte and characteristic polynomials.

$$\left(\frac{5}{x^5}+\frac{10}{x^3}+\frac{8}{x}+9x+3x^3\right)y+\left(\frac{3}{x^3}+\frac{7}{x}+5x+5x^3+x^5\right)y^3+ \\ +\left(x+3x^3+2x^5+x^7\right)y^5+x^9y^7 \quad (5.1)$$

The Tutte polynomials of the two graphs are quite large so instead we show their chromatic polynomials, which are evaluations of the respective Tutte polynomials:

$$-12x+36x^2-43x^3+26x^4-8x^5+x^6$$
$$-14x+39x^2-44x^3+26x^4-8x^5+x^6$$

The characteristic polynomials are:

$$-8x+6x^2+17x^3-4x^4-9x^5+x^7$$
$$2-10x+4x^2+17x^3-4x^4-9x^5+x^7$$

Thus we can neither find the spectrum nor the Tutte polynomial from the Ising polynomial. In fact, by two slightly denser graphs, shown in Figure 4, we can also find isomagnetic graphs with different chromatic numbers and different clique numbers. Recall that by Theorem 4.3 the latter is determined by the Ising polynomial if G is regular.

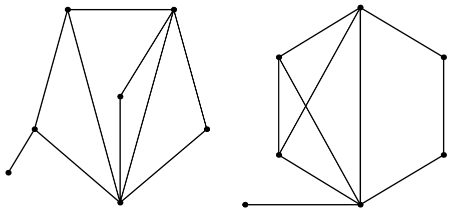

FIGURE 4. Isomagnetic graphs with different clique numbers.

6. Families of isomagnetic graphs

Our next aim is to prove that there exists infinitely many graphs which are not magnetically unique, and also that there exists arbitrarily large sets of pairwise isomagnetic graphs. Our construction will be an adaption of the so called rotor graph construction used by Tutte in connection with the Tutte polynomial. See Chapter 6 of [Tut98] for a very enjoyable discussion of this type of construction. Using this construction we will be able to find large sets of isomagnetic planar triangulations.

In Figure 5 we show a graph which we will call the Rotor graph R. Note that R has three-fold rotational symmetry but no other automorphisms. Analogously with Tutte's construction we now have the following simple

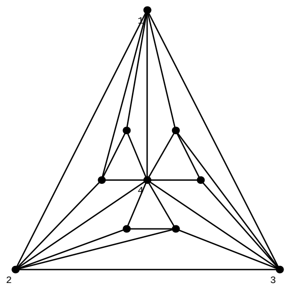

FIGURE 5. The rotor gadget R.

lemma.

Lemma 6.1. *Let G be a planar graph which contains R as an induced subgraph, and let G' be the graph obtained from G by replacing R by its reflection in the line through vertices 1 and 4. Then G and G' have the same Ising polynomial.*

Proof. This follows from the simple observation that we have, via reflection, a bijection between states in G and G', which preserves the energy and magnetisation of the states. □

This can also be expressed using the partial state expression for the Ising polynomial, with H as the graph induced by vertices $1, 2, 3$.

Next we will need two theorems regarding the properties of uniform random planar triangulations.

Theorem 6.2 ([GW04]). *Let M be a planar 3-connected triangulation and let X_n be the random variable counting the number of copies of M in a random 3-connected triangulation on n vertices.*

Then there exists constants A_M and B_M such that $\frac{X_n - nA_M}{\sqrt{nB_m}}$ converges in distribution to the standard normal random variable, as $n \to \infty$.

Theorem 6.3 ([RW95]). *Let T be a random planar 3-connected triangulation on m edges. Then the probability that T has a nontrivial automorphism is less than C^m, for some constant $0 < C < 1$.*

We can now prove that the large equivalence classes we want actually exists.

Theorem 6.4. *For every N there exists a family of at least N non-isomorphic isomagnetic graphs.*

Proof. By Theorem 6.2 we can find a graph G which contains at least $\log_2 N$ edge disjoint copies of R, since any copy of R can only overlap with a bounded number of other copies. By Theorem 6.3 we can also find such a G with a trivial automorphism group. Let X be the set of graphs obtained by applying the operation in Lemma 6.1 to every subset of copies of M in G. By the previous observations the graphs in X are non-isomorphic and have the same Ising polynomial, and X has cardinality at least N. □

In fact, as described in [Tut98], these graph families will also have the same Tutte polynomial. An obvious question is if there is any natural graph invariant which separates the graphs in one of these families? As the proof actually shows that for most triangulations we can find some non-zero number of iso magnetic graphs we also have this corollary

Corollary 6.5. *There exists a constant $0 < D < 1$ such that the probability that a random planar 3-connected triangulation on n vertices is magnetically unique is less than D^n.*

We also conjecture that

Conjecture 6.6. *Given any $c > 0$ and $\frac{c}{n} < p(n) \leq 1/2$ the probability that a graph from $G(n, p(n))$ is magnetically unique is less than D^n, for some $D < 1$.*

Using the examples of small isomagnetic trees this conjecture could be approached for $p(n) < \frac{1}{n}$, but for larger p something better is needed.

7. Conclusions

The multivariate Ising polynomial and the Tutte-polynomial share a lot of information since they both contain the partition function of the $q = 2$ random cluster model, which is equivalent to $Z(G, x, 1)$. However as our examples show they are also clearly distinct, as witnessed by the fact that the Ising polynomial is a fairly strong invariant for trees and the Tutte polynomial is identical for trees of the same order. Finding further similarities and differences between the two polynomials is an interesting area for further research.

Another contrast between these two polynomials is that the Tutte polynomial is easily extended to a polynomial for matroids rather than just graphs, as discussed in e.g. [Sok05]. For the Ising polynomials this matroid connection comes from the cut definition of the Ising polynomial, which is easy to generalise to a matroid when $y = 1$. However, for a general matroid we do not have a natural generalisation of the number of vertices on each side of a cut, and thereby we do not know how to define the polynomial for a general y.

For the van der Waerden polynomial the situation is slightly better. We do not know what the best generalisation of $W(G, t, u)$ to matroids will be, but we have at least one possibility. Let M be a matroid with base set $E(M)$ and let G be a subset of $E(M)$. For a subset H of G define

$$d(H, M) = \min_{y \in \mathcal{C}(M)} |H \triangle y|,$$

where $A \triangle B$ denotes the symmetric difference of the two sets A and B, and $\mathcal{C}(M)$ is the set of circuits of the matroid M. We can now define

$$W(M, G, t, u) = \sum_{H \subset G} t^{|H|} u^{2d(H,M)}.$$

If we take M to be the cycle matroid K_n and G as the set of edges of a graph on n vertices this coincides with our original definition of the van der Waerden polynomial for a graph G. Whether this polynomial has an interesting structure for other matroids remains to be seen.

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