Extremal graphs for some problems on cycles in graphs

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Abstract. This paper contain a collection of extremal graphs for some questions on cycles in graphs. The graphs have been found by exhaustive computer search.

I list the extremal graphs and values for the maximum and minum numer of cycles in a graph, graphs without cycles of length 4 and 8 relating to a conjecture of Erdös and Gyarfas and the smallest 3-connected non-hamiltonian cubic graphs of class I.

1. Introduction

Many basic questions regarding the cycle structure of graphs in general, and cubic graphs in particular, are very poorly understood. In the literature we find unsolved problems of all degree of sophistication, from the cycle double cover conjecture to the hamiltonicity of various classes of graphs.

In this paper I list extremal graphs and values for some graph properties related to cycles in graphs which I have happend to be working on during the last few years. Next to more traditional pen and paper work I have done some computational work which has been collected in this note.

The four different problems treated are briefly: the maximum number of cycles, the minimum number of cycles, the existence of cycles of length a power of 2, and finding nonhamiltonian 3-connected, 3-edge colourable cubic graphs.

2. Graphs with many cycles

In [ES81] Entringer and Slater studied graphs with the maximum possible number of cycles among all graphs on n vertices and m edges. More specifically they defined $\psi(G)$ to be the number of cycles in the graph G and $\psi(k)$ as the maximum number of cycles in a graph with $n+k-1$ edges. Furthermore they showed that given any value of k there is a cubic graph G on $2(k-1)$ vertices such that $\psi(G)=\psi(k)$, i.e. there is always a reasonably small extremal graph for a given k.

Since k is the dimension of the cycle space of a graph on $n+k-1$ edges we find that $\psi(k) < 2^k$. Entringer and Slater proved that $\psi(k) \geq$

$2^{k-1} + k^2 - 3k + 3$ by calculating the number of cycles in the Möbius wheels. Using an exhaustive computer search they found the value of $\psi(k)$ for $k \le 8$ and based on these values conjectured that $\psi(k) \sim 2^{k-1}$.

I have extended the computer search for cubic graphs on $2(k-1)$ vertices for which $\psi(G) = \psi(k)$. I have found the extremal graphs for $k \le 11$. For $12 \le k \le 22$ I have computed lower bounds for $\psi(k)$ by narrowing our search to graphs with high girth. The results are given in Table 1.

In [ES81] it was conjectured that all cubic graphs which are extremal for ψ would have as large girth as is possible for a cubic graph on $2(k-1)$. This conjecture was disproved in [Gui96], however it still seems to be true that the extremal graphs tend to have a girth which is close to the largest possible one. Thus it is not unreasonable to expect the lower bounds for $\psi(k)$ given here to actually be the value of $\psi(k)$.

The generalised Petersen graph $\mathrm{GP}(n,m)$, $n \ge 3$, $1 \le m < n/2$, is a cubic graph with vertex-set $\{u_i; i \in Z_n\} \cup \{v_i; i \in Z_n\}$, and edge-set

$$\{u_i u_{i+1}, u_i v_i, v_i v_{i+m}; i \in Z_n\}.$$

For $k = 6$ we have seen that $GP(6-1,2)$, the ordinary Petersen graph, is extremal with respect to $\psi(G)$, likewise for $k = 5$ the graph $GP(5-1,2)$, the cube, is extremal. Motivated by these to initial coincidences we computed the number of cycles in all small generalised Petersen graphs and found that for suitable m they come very close indeed to the extremal value of $\psi(n-1)$, see Table 2. We thus make the following conjecture.

Conjecture 2.1. *Let* $p(k) = \max_m \psi(GP(k-1,m))$.

(1) $p(k) = 2^{k-1} + f(k)$, *where* $f(k)$ *is a function not bounded by any polynomial.*
(2) $\psi(k) - 2^{k-1} = \mathcal{O}(p(k) - 2^{k-1})$.

From our data for small k one find a decent fit with $f(k) = \mathcal{O}(k^{\ln k})$. However there is little else in support for a sharper conjecture.

That $p(k)$ is greater than 2^{k-1} is immediate since $GP(k-1,1)$ is the family of ordinary cyclic ladders and for these we have that

$$\psi(GP(k-1,2)) \ge 2^{k-1} + (k-1)(k-2) + 2.$$

k	$\psi(k)$	$\psi(k)/2^{k-1}$	girth	Extremal graphs
3	7	1.75	3/3	K_4, 3-cage
4	15	1.875	3/4	$K_{3,3}$, 4-cage
5	29	1.813	3/4	The Möbius ladder.
6	57	1.7813	3/5	The Petersen graph, 5-cage
7	109	1.703	3/5	Figure 1
8	213	1.664	3/6	The Heawood graph, 6-cage
9	401	1.566	3/5	Figure 2
10	783	1.529	3/6	Figure 2
11	1484	1.449	3/5,6	Figure 3
12	2876	1.404	3/6	Figure 4
13	$\geq$ 5608	$\geq$1.369	6/7	The McGee graph, 7-cage
14	$\geq$ 10872	$\geq$1.327	7/7	Figure 4
15	$\geq$ 21192	$\geq$1.293	7/7	The Coxeter graph, snark
16	$\geq$ 41400	$\geq$1.263	7/8	The Tutte-Coxeter graph, 8-cage
17	$\geq$ 80211	$\geq$1.223	7/7	Figure 5
18	$\geq$ 157134	$\geq$1.198	8/8	Figure 5
19	$\geq$ 306373	$\geq$1.168	8/8	Figure 6
20	$\geq$ 600054	$\geq$1.144	8/8	Figure 6
21	$\geq$ 1182592	$\geq$1.127	8/8	Figure 7
22	$\geq$ 2324532	$\geq$1.108	8/8	Figure 7

TABLE 1. Values and bounds for $\psi(k)$. In the column labelled girt the first value is the minimum girth of the graphs searched to give our bound and after the slash mark we list the girths of the extremal graphs found.

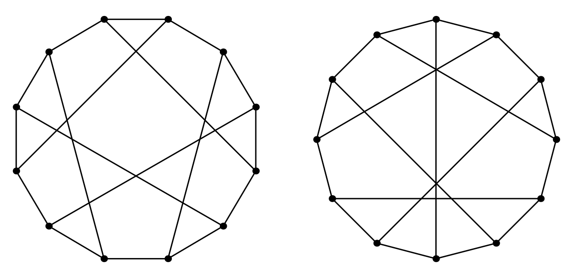

FIGURE 1. Graphs maximising $\psi(G)$ for $k = 12$.

k	$\psi(k)$	$p(k)$
4	15	14
5	29	28
6	57	57
7	109	94
8	213	205
9	401	400
10	783	704
11	1484	1456
12	2876	2818
13	$\geq$ 5608	5442
14	$\geq$ 10872	10818
15	$\geq$ 21192	19767
16	$\geq$ 41400	40099
17	$\geq$ 80211	72656

TABLE 2. Values of $p(k)$ for $k \leq 17$ compared to our best lower bounds for $\psi(k)$

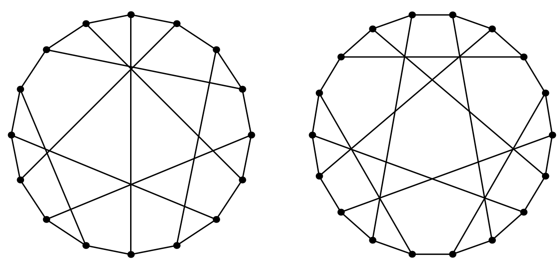

FIGURE 2. Graphs maximising $\psi(G)$ for $k = 16, 18$.

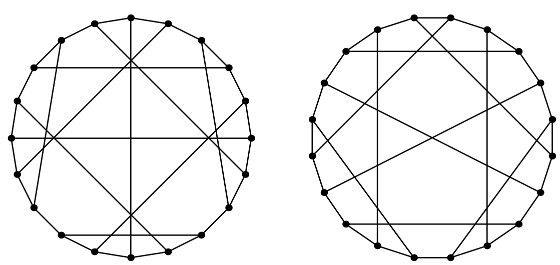

FIGURE 3. Graphs maximising $\psi(G)$ for $k = 20$.

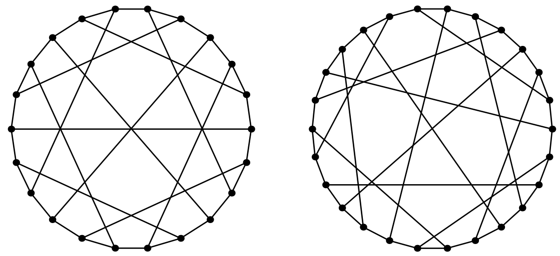

FIGURE 4. Graphs giving our value and lower bound for $\psi(k)$ for $k = 22, 26$.

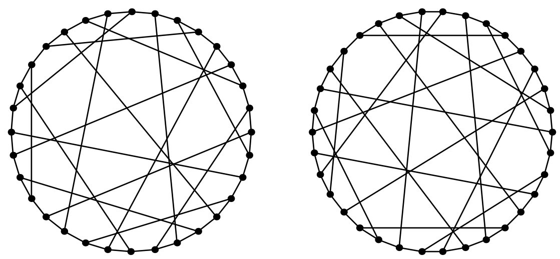

FIGURE 5. Graphs giving our lower bound for $\psi(k)$ for $k = 32, 34$.

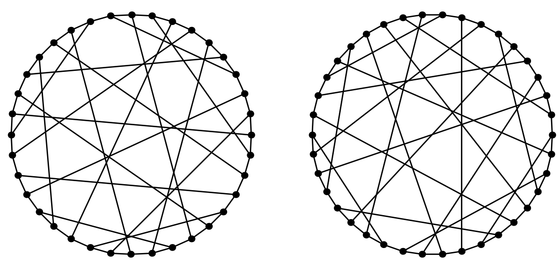

FIGURE 6. Graphs giving our lower bound for $\psi(k)$ for $k = 36, 38$.

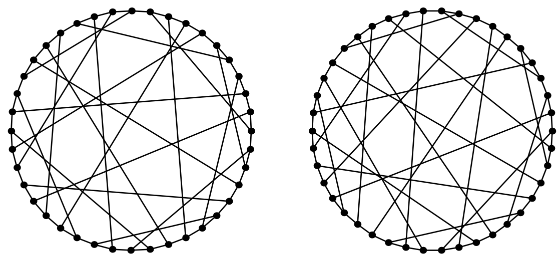

FIGURE 7. Graphs giving our lower bound for $\psi(k)$ for $k = 40, 42$.

3. Graphs with few cycles

In [BCE86] Barefoot Clark and Entringer studied what is more or less to opposite question of the previous section, namely how many cycles a graph on n at least must have. Since trees by definition do not have any cycles we must restrict our graph in order to get a meaningful question. Barefoot et al defined $f_k(n)$ as the smallest number of cycles in any k-connected cubic graph on n vertices.

Barefoot et al determined the exact value of $f_1(n)$ and found the family of extremal graphs satisfying this bound, f_1 turns out to be linear in n. They also found the value $f_2(n)$, which is quadratic in n, an found the corresponding extremal graphs.

For $k = 3$ $f_k(n)$ has not yet been determined. In [BCE86] it was shown that $f_3(n)$ is bounded from above by $n2^{n^c}, c = \frac{\log 8}{\log 9}$, and it was conjectured that $f_3(n)$ is larger than any polynomial. This conjecture was proved by Aldred and Thomassen [AT97] who showed that $2^{n^{0.17}} < f_3(n)$ for large n. These bounds are to this date the best found for $f_3(n)$.

Obviously no family of extremal graphs for $f_3(n)$ has been found, but in [BCE86] the extremal graphs for $n \leq 14$ was found by comparing a simple family of planar graphs with values of $\psi(G)$ from a computer generated table. The family $\mathcal{G}_n$ mentioned is constructed as follows. Let $\mathcal{G}_n = \{K_4\}$. A graph $G_2 \in \mathcal{G}_{n+1}$ is constructed from a graph G_1 in $\mathcal{G}_n$ by expanding a vertex v, contained in as few cycles of G_1 as possible, into a triangle. It was also noted without proof that if $G \in \mathcal{G}_n$ then $\psi(G) = F_{n/2+5} - n/2 + 4$, where F_n is the nth Fibonacci number.

We have verified and extended the search for extremal graphs for $f_3(n)$ to all $n \leq 20$ and we found that all the extremal graphs thus far belong to $\mathcal{G}_n$. As noted in [BCE86] this can not continue to hold for all n, the Fibonacci number grow exponentially, and an interesting challenge is thus to find the smallest n for such that $f_3(n) < F_{n/2+5} - n/2 + 4$.

The extremal graphs for $n \leq 20$ are displayed in Figures 8 to 13.

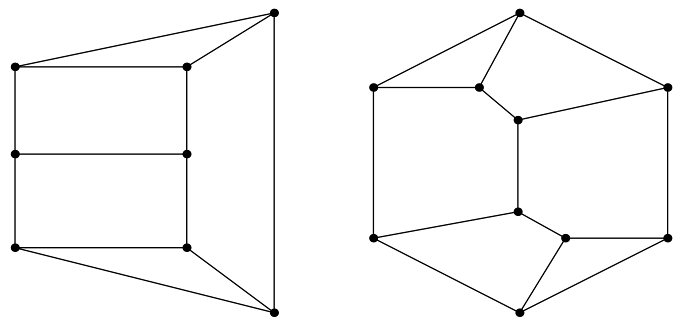

Figure 8. Graphs minimising $\psi(G)$ for $n = 8, 10$.

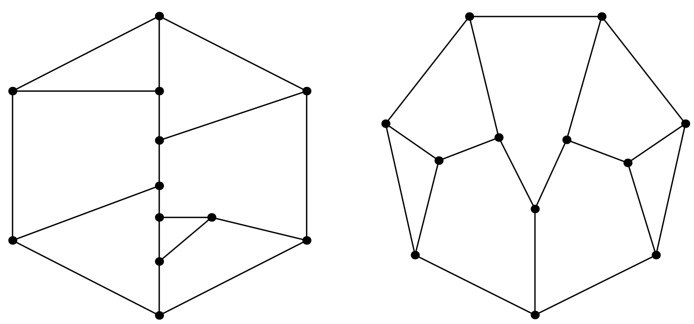

Figure 9. Graphs minimising $\psi(G)$ for $n = 12$.

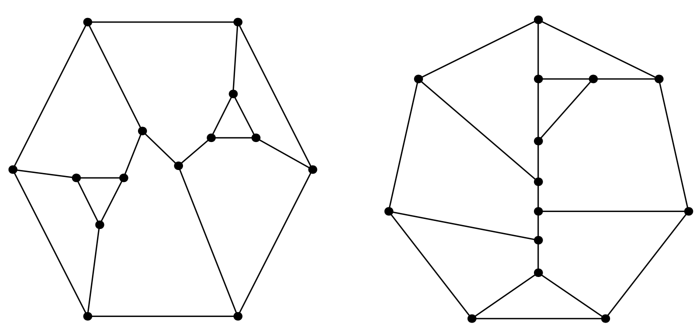

Figure 10. Graphs minimising $\psi(G)$ for $n = 14$.

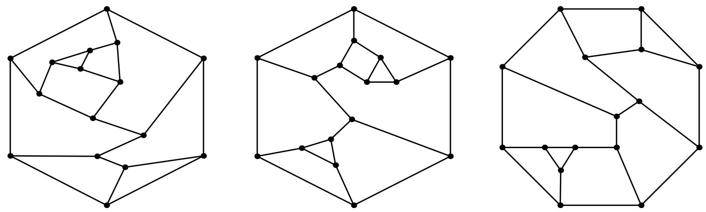

FIGURE 11. Graphs minimising $\psi(G)$ for $n = 16$.

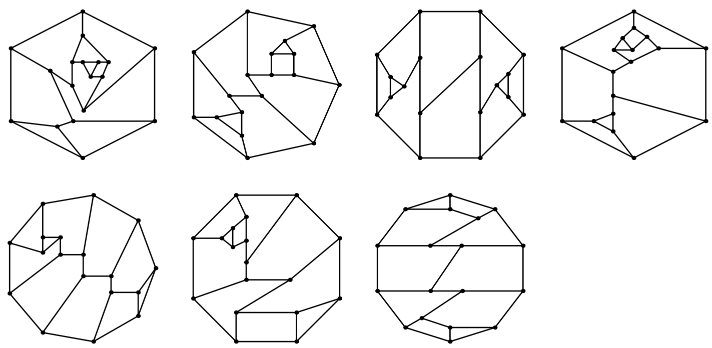

FIGURE 12. Graphs minimising $\psi(G)$ for $n = 18$.

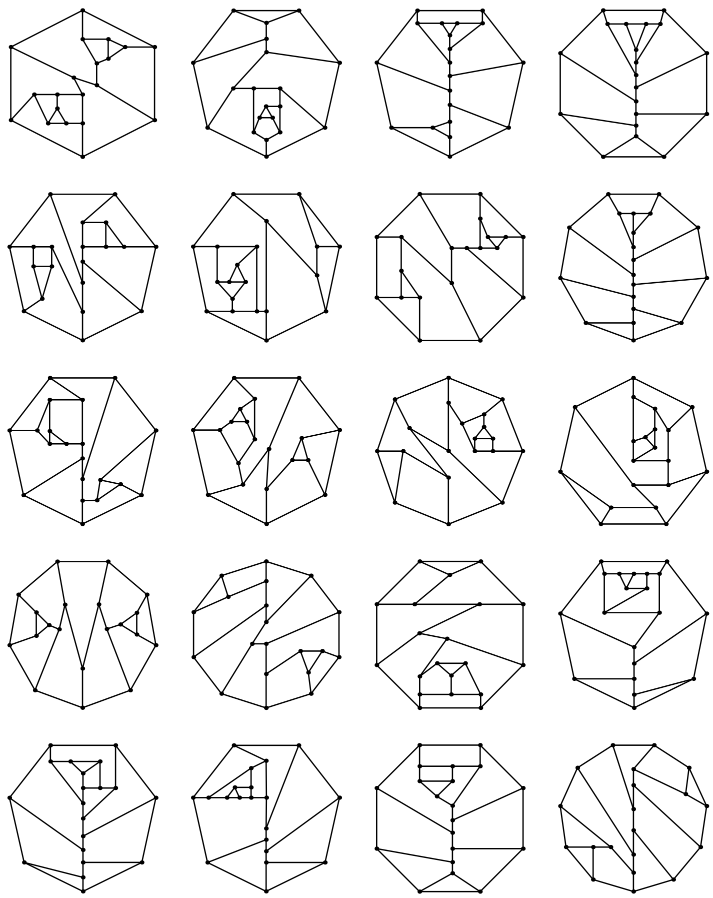

FIGURE 13. Graphs minimising $\psi(G)$ for $n = 20$.

n	
24	4
26	23
28	251

TABLE 3. The number of cubic graphs with no C_4 and C_8.

4. Graphs with few cycles of length 2^k

In 1995 Erdös and Gyarfas made the following conjecture.

Conjecture 4.1. *Every graph with minimum degree at least 3 contains a cycle whose length is a power of 2.*

Erdös offered $100 for a proof and $50 for a counterexample. Apart from Shauger's results on claw-free graphs [Sha98, DS01] there seems to be very little published on this conjecture.

Assume that G is a, edge and vertex, minimal counterexample to this conjecture and that u and v are two vertices of G. If $d(u) \geq 3$ and $d(v) \geq 3$ then $\{u, v\}$ can not be an edge; if it was then $G \setminus \{u, v\}$ would be a counterexample with fewer edges than G. Thus a counterexample must consist of an independent set V_1 of vertices of degree at least 4, and a nonempty set $V_2 = V \setminus V_1$ of vertices of degree 3.

Using this observation Gordon Royle [Roy] used a modified version of Brendan McKay's graph generator `makeg` [McK] to generate graphs without C_4's and the described degree structure. Royle generated all relevant graphs on less than 16 vertices and found no counterexamples.

In order to extend this search further we choose to look at graphs with $V_1 = \emptyset$, i.e cubic graphs. We used Gunnar Brinkman's cubic graph generator minibaum [Bri96] to generate all cubic graphs on less than 29 vertices and a simple fortran program to check for the existence of cycles of length 4,8 and 16. No counterexamples to the conjecture was found. However on 24 vertices we found the smallest cubic graphs without cycles of lengths 4 and 8. These graphs are displayed in Fig. 14. We note that the lower right of the four graphs can be constructed from K_4 be repeatedly expanding vertices into triangles, it is also the only planar graph among the four.

In Table 3 we display the number of cubic graphs on n vertices having no cycles of length 4 or 8. From the distribution of short cycles in random cubic graphs, see e.g. the nice survey in [Wor99], we find that for these small n the number of graphs without any cycles of length 4 and 8 is much larger than the asymptotic proportion of such graphs among the cubic graphs.

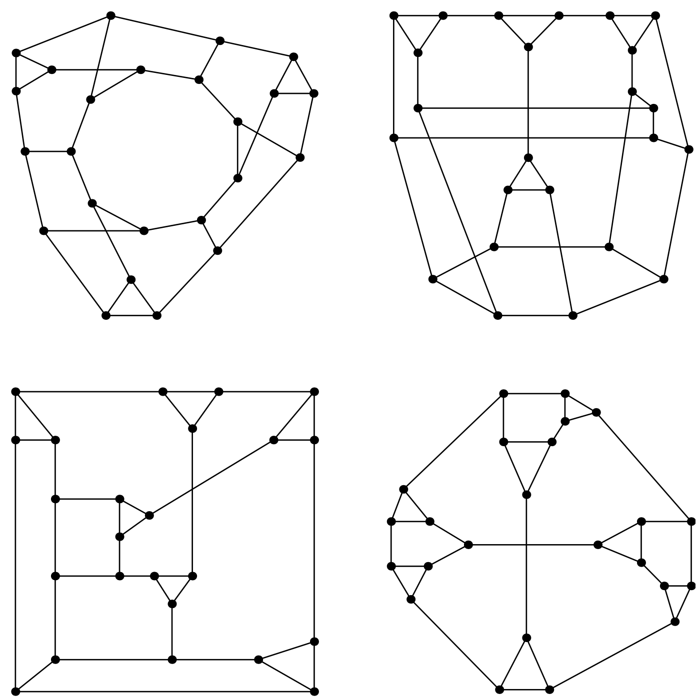

FIGURE 14. The cubic graphs on 24 which contains neither a C_4 nor a C_8

5. Nonhamiltonian cubic graphs of class I

The hamiltonicity of cubic graphs is a well investigated property. It is of course easy to construct a nonhamiltonian 1-connected cubic graphs and it is only slightly harder to find a 2-connected nonhamiltonian graph. Among the 3-connected cubic graphs the smallest nonahmiltonian graph is the well known Petersen-graph, and from it there is a number of ways to construct families of larger nonhamiltonian graphs. Graphs constructed in this way are typically of class II, i.e. they are not 3-edge-colourable.

Among cubic graphs of class I the planar graphs have received a lot of attention, first in connection with the 4-colour theorem. The first non-hamiltonian example was found by Tutte in [Tut46]. Later on several examples of non-hamiltonian cubic graphs on 38 vertices were found by Lederberg, Barnette, and Bosk. Holton and McKay proved that all 3-connected planar cubic graphs on at most 36 vertices are hamiltonian, thus proving the minimality of the examples on 38 vertices.

Here we have searched for 3-connected, non-planar, non-hamiltonian, cubic graphs of class I. We used a fortran program to sieve for non-hamiltonian graphs among all cubic graphs on n vertices and among those we used Mathematica to check wether the found graphs were class I and 3-connected. The cubic graphs were generated using minibaum [Bri96]. All triangle free cubic graphs on at most 22 vertices were tested, the restriction to triangle free graphs coming from the fact that if we contract a triangle in a cubic graph we preserve both the chromatic index and the hamiltonicity of the graph.

In Figure 15 we show the smallest such graphs and in Figure 16 we show all such graphs on 22 vertices. The first four graphs in Figure 16 can be constructed from the graph in Figure 15, the fifth graph is the generalised Petersen graph $GP(11, 2)$.

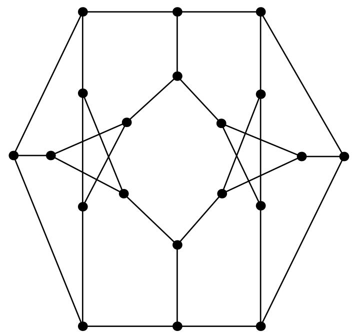

FIGURE 15. The 3-connected class-I nonhamiltonian cubic graph on 20 vertices.

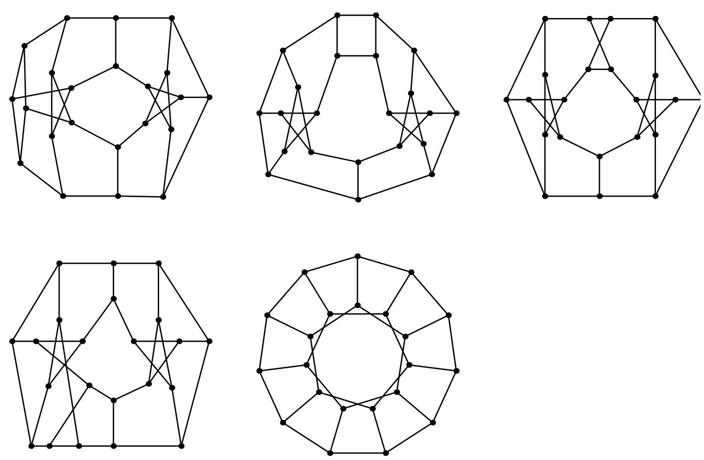

FIGURE 16

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