Convexification of Pseudoconvex Domains

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Abstract. In 1976 Fornaess [For76] proved that any strictly pseudoconvex domain in $\mathbb{C}^n$ can be embedded into a strictly convex domain in $\mathbb{C}^m$, for some $m \geq n$. In this paper we study the properties of domains which can be embedded into a convex domain and give further examples of classes of pseudoconvex domains with such embeddings.

1. Introduction

One of the main differences between complex analysis in one and several variables is the crucial role that the exact geometric properties of different domains begin to play. In one variable the Riemann mapping theorem tells us that there is basically one simply connected domain to study, the unit disc, and that the multiply connected cases are only marginally more complicated. In several variables there are no such simplifying circumstances and one can in fact find domains which are arbitrarily close to the unit ball and are still not biholomorphic to the ball.

In this plethora of domains and their associated sets of holomorphic functions we find that many techniques that we could use for the one variable case no longer work. Among other things we get into trouble due to the unwieldy nature of zero-sets of even simple functions like polynomials which no longer are mere point sets but rather some large and and often uncontrollably winding variety.

However there are still some domains on which things work in a fairly simple way, notably the geometrically convex sets and also the wider class of strictly pseudoconvex sets. On geometrically convex sets we can in a simple way deduce properties like polynomial convexity and the existence of Stein neighbourhood bases thanks to simple geometrical constructions.

Due to the aforementioned lack of biholomorphisms we can not hope to exhaust a particularly large class of domains thanks their equivalence to convex domains. From a well known theorem by Narashiman (see [Kra92]) we do however know that any part of the boundary of a strictly pseudoconvex set can be made convex with a local biholomorphism. The problem is that in most cases this biholomorphism can not be extended to make the rest of the domain convex as well, for a further discussion of this see [Noe91]. But all is not lost, in [For76] Fornaess proved for strictly pseudoconvex domains that while we can not hope to find biholomorphisms to convex domains of the same dimension as our original domain we can always find a biholomorphic embedding of it into a convex domain in some higher dimension. Biholomorphic embeddings are not quite as good as ordinary biholomorphisms since some biholomorphically invariant properties are not invariant when we change dimension, but many properties are still invariant and we can make good use of our embedding.

In this paper we study some of the properties of domains which can be biholomorphically embedded into convex domains and give some new examples of classes of domains which admit such embeddings.

2. Basic definitions

At first the theorem by Fornaess mentioned in the introduction can seem a bit perplexing since it among other things tells us that many non-simply connected domains can be given convex embeddings. In order to show how this is possible and give us something concrete to have in mind later on we will start this section by an example.

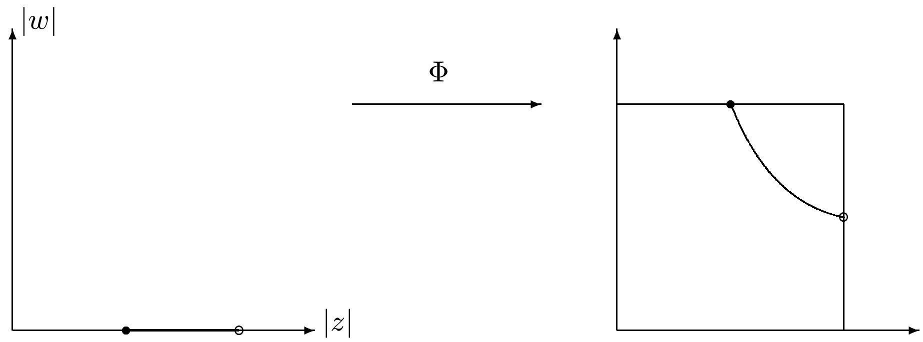

FIGURE 1. A modulus-plane view of the embedding in Ex 2.1

Example 2.1. Let U be an annulus in $\mathbb{C}$ centred at 0 having outer radius 1 and inner radius $\frac{1}{2}$. Now define a mapping from U into the bi-disc D as $\Phi(z) = (z, \frac{1}{2z})$. The mapping is a biholomorphic embedding of U into the convex domain D just like we wanted. In Fig.2.1 we can see the simple geometrical idea behinds this. The important thing to notice here is that Φ gives a proper embedding of U into D, that is, the boundary is mapped into the new boundary and the interior into the interior.

We will now start to formalise some of the properties of Ex.2.1. First some general notation.

Definition 2.2.

(1) Given an domain Ω let $H(\Omega)$ denote the set of functions which are holomorphic in Ω.

(2) Let $A^k(\Omega), k \in \{0, 1, 2, \dots, \infty, \omega\}$, denote $C^k(\overline{\Omega}) \cap H(\Omega)$. Here A^ω denotes the space of functions which are holomorphic on some neighbourhood of $\overline{\Omega}$.

(3) Given two domains $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^m, m \geq n$, we let $B^k(\Omega_1, \Omega_2)$ denote the set of proper bijective holomorphic mappings from Ω_1 into Ω_2 which extend to C^k

diffeomorphisms from $\bar{\Omega}_1$ to $\bar{\Omega}_2$. We let $B^\omega(\Omega_1, \Omega_2)$ denote the space of mappings which extend to a biholomorphism of a neighbourhood of $\bar{\Omega}_1$.

In order to get a useful embedding we need put some conditions on it. First we want our mapping to be proper, that is we want it to map the boundary of the first domain, Ω_1, into the boundary of the second domain, Ω_2 , and the interior into the interior. Second we also want to make sure that in the case that our mapping extends to a domain larger than Ω_1 it takes the exterior of Ω_1 into the exterior of Ω_2. These conditions will make sure that we can use many of the properties of the convex domain that we have embedded Ω_1 into. The reader can easily check that these conditions hold in Ex.2.1.

We now make the formal definition of our main concept and in order to avoid having to repeat the phrase 'possible to embed into a convex domain by a C^k-mapping, we introduce the term *convexifiable* as follows:

Definition 2.3.

(1) A domain $\Omega \subset\subset \mathbb{C}^n$ is said to be *convexifiable* if there is a convex domain $\widehat{\Omega} \subset\subset \mathbb{C}^m, m \geq n$, and a proper, bijective holomorphic embedding $\phi : \overline{\Omega} \to \overline{\widehat{\Omega}}$, $\phi(z) = (\phi_1(z), \phi_2(z), \ldots, \phi_m)$, $\phi_j \in C(\overline{\Omega})$.

Any such set $\widehat{\Omega}$ is said to be an *inflation* of Ω_1.

(2) Ω is said to be *strictly convexifiable* if $\widehat{\Omega}$ can be chosen to be geometrically strictly convex. A domain Ω is said to be geometrically strictly convex if any line connecting two points on the boundary of Ω does not intersect the boundary in any other points except these two.

(3) Ω is said to be *k-convexifiable*, $k \in \{1, 2, \ldots, \infty\}$, if ϕ can be chosen from $B^k(\Omega, \widehat{\Omega})$, and *$\omega$-convexifiable* if the mapping belongs to $B^\omega(\Omega, \widehat{\Omega})$ and the mapping ϕ takes the exterior of Ω into the exterior of $\widehat{\Omega}$.

In this terminology Ex 2.1 shows that the annulus is an ω-convexifiable domain. Using the same mapping it can actually be seen to be strictly ω-convexifiable with the ball as an inflation.

Here it is also important to notice that we demand that the mapping ϕ must be continuous on the closure of Ω and not just on in the interior. To see the importance of this it can be useful to keep the following example in mind as we go along.

Example 2.4. Let Ω be the Hartogs triangle $\Omega = \{(z, w) : \ |z| < |w| < 1\}$ and let ϕ be the mapping $\phi(z, w) = (z, w, \frac{z}{w})$. This mapping maps Ω into the polydisc in $\mathbb{C}^3$ but is not continuous at the origin. The Hartogs triangle is a simple example of a domain which is not a domain of existence for A^∞ and does not have a Stein neighbourhood basis.

3. Some properties of convexifiable domains

In this section we will take a look at some properties of domains which are possible to embed into a convex domain. We will see that they share some of the nice properties of convex domains and at the end of the section we will also mention some cases where things do not work out, or at least are not as simple as one might have hoped for.

Observation 3.1. *A convexifiable domain Ω is pseudoconvex.*

Proof. Let $\widehat{\Omega}$ be an inflation of Ω and ϕ the mapping connecting them.

Given any point $p \in \partial\widehat{\Omega} \cap \phi(\Omega)$ we can, since $\widehat{\Omega}$ is convex, find a function $f \in A^0(\widehat{\Omega})$ such that $p \in \mathcal{Z} = \{z : f(z) = 0\}$ and $\mathcal{Z} \cap \widehat{\Omega}^\circ = \emptyset$. Thus the function $g = (f \circ \phi)^{-1}$ is a function in $H(\Omega)$ which cannot be extended over $\phi^{-1}(p)$. $\square$

If the domain is ω-convexifiable it is not only pseudoconvex but can also be written as the intersection of pseudoconvex domains in a very nice way.

Theorem 3.2. *If Ω is ω-convexifiable then $\bar{\Omega}$ has a Stein neighbourhood basis. More precisely there exists a continuum of domains W_t, $0 \leq t \leq 1$, such that*

(1) $W_0 = \Omega$.

(2) $W_i \subset\subset W_j$ *if* $i < j$.

(3) ∂W_t *is homotopic to* $\partial\Omega$.

(4) W_i *is Runge with respect to* W_j *if* $i < j$.

Proof. Let $\widehat{\Omega}$ be an inflation of Ω, without loss of generality we can assume that 0 lies in the interior of $\widehat{\Omega}$, and ϕ the mapping from Ω into $\widehat{\Omega}$.

Let ψ_k be defined by $z \mapsto kz$, for $k > 1$. Clearly ψ_k is a biholomorphism of $\widehat{\Omega}$ onto a scaled copy of itself and since ψ_k fixes the origin $\widehat{\Omega} \subset\subset \psi_k(\widehat{\Omega})$.

Now let Ω_2 be a domain such that $\Omega \subset\subset \Omega_2$ and ϕ is bijective from Ω_2 to $\phi(\Omega_2)$. Let $c > 0$ be a number such that $\phi(\partial\Omega_2)$ lies outside $\psi_{1+c}(\widehat{\Omega})$.

Define $W_t = \phi^{-1}(\phi(\Omega_2) \cap \psi_{1+ct}(\widehat{\Omega}))$. Properties 1 to 3 in the theorem can easily be verified and property 4 follows from paragraph 4 in [DG60]. □

Our next theorem concerns the boundary structure of a convexifiable domain. We recall that a point p is called a *peak-point* for a function space $\mathcal{A}(\Omega)$ if there exists a function $f \in \mathcal{A}(\Omega)$ such that $|f(p)| = \sup_{z\in\overline{\Omega}} |f(z)|$, and $|f(z)| < |f(p)|$ if $z \in \overline{\Omega}$, $z \neq p$.

Theorem 3.3. *If Ω is a strictly k-convexifiable domain then every boundary point is a peak-point for $A^k(\Omega)$.*

Proof. Let $\widehat{\Omega}$ be an inflation of Ω and Φ the map from Ω into $\widehat{\Omega}$.

Now let $p \in \partial\Omega$ and W be a hyperplane, defined by $F(z) = 0$, where F is a linear function in z, not intersecting $\widehat{\Omega}$ such that $\Phi(p)$ is the unique point in $\overline{\widehat{\Omega}}$ at minimum distance from W. Such a W exists since $\widehat{\Omega}$ is strictly convex.

Set $f_p(z) = \frac{1}{F(\Phi(z))}$. It is easily checked that $|f(z)|$ attains its maximum at p and belongs to $A^k(\Omega)$ $\square$

Together with the following result of Basener [Bas77] we obtain an interesting property of strictly convexifiable domains.

Lemma 3.4 (Basener)**.** *Let Ω be a domain with C^2 boundary and p a boundary point which is a peak point for $A(\Omega)$, then p is the limit of a set of strictly pseudoconvex boundary points.*

Corollary 3.5. *If Ω is a strictly convexifiable domain with C^2-boundary then the set of strictly pseudoconvex boundary points of Ω is dense in $\partial\Omega$.*

Proof. The result follows directly from the above theorem and Lemma 3.4. $\square$

We now turn to some questions about approximation of functions on Ω and the number of generators of some of the function algebras on Ω. Let $\mathcal{A}(\Omega)$ be a Banach algebra of holomorphic functions on Ω. A set of functions $\{f_1, f_2, \ldots, f_k\}, f_i \in \mathcal{A}(\Omega)$, is said to generate $\mathcal{A}(\Omega)$ topologically if the $\mathcal{A}$-norm closure of the set of polynomials in f_i is $\mathcal{A}(\Omega)$ and the f_i are called *generators* for $\mathcal{A}(\Omega)$.

Furthermore we say that a boundary point $p \in \partial\Omega$ is a *strictly convex* boundary point of Ω if there is a tangent hyperplane P of $\partial\Omega$ such that $P \cap \partial\Omega = p$. Observe that this property is not local.

Theorem 3.6. *If Ω is k-convexifiable then $H(\Omega)$ is generated, in the sense of uniform convergence on compact sets, by a finite set of functions in $A^k(\Omega)$.*

Proof. Let $\widehat{\Omega}$ be an inflation of Ω and ϕ the mapping between them.

Every function $f \in H(\Omega)$ defines a holomorphic function $f_1 = f \circ \phi^{-1}$ on $\phi(\Omega)$. By theorem I5 in Gunning [Gun90a] we get that f_1 can be extended to a holomorphic function $f_2 \in H(\widehat{\Omega})$.

Since $\widehat{\Omega}$ is convex its closure is also polynomially convex and thus every function in $H(\widehat{\Omega})$ can be approximated uniformly on compact subsets by polynomials.

This also means that f_2 can be approximated uniformly on compacts in $\phi(\Omega)$ by polynomials. Thus every function in $H(\Omega)$ can be approximated on compacts in Ω by polynomials in the components of the mapping ϕ and the components of Φ is thus a set of generators for $H(\Omega)$.

□

From our earlier results we also have the following.

Corollary 3.7. *If Ω is k-convexifiable then $\bar{\Omega}$ is a domain of existence for $A^k(\Omega)$.*

Proof. Given $p \in \partial\Omega$ let f be a (weak) peak function for p, as guaranteed by Theorem 3.3, and choose f such that $f(p) = 1$. Then the function $(1 - f(z))^{k+1} \sin((1 - f(z))^{-1})$ will be a function in $A^k(\Omega)$ which does not extend over p. □

This corollary does in fact imply not only that Ω is a domain of existence but also that any analytic subvariety of Ω is a domain of existence for A^k. For an example of a domain of existence which does not have this stronger property see [BF98]. Domains which do not have the stronger property are of some interest in their own since they give, at least as far as the author knows, the only known way to explicitly construct homomorphisms in the corona of $A^k(\Omega)$.

Next we will return to another question regarding generators, the Gleason-property. Let $\mathcal{A}$ be an algebra of functions on Ω and let $p \in \Omega$. We say that Ω has the Gleason

$\mathcal{A}$-property at p if the maximal ideal of functions vanishing at p is algebraically generated by the coordinate functions $z_1 - p_1, z_2 - p_2, \ldots, z_n - p_n$. That is every function in this ideal should be expressible as a polynomial in the coordinate functions with coefficients from $\mathcal{A}$. If Ω has the Gleason-property at every $p \in \Omega$ then Ω is said to have the Gleason $\mathcal{A}$-property.

Furthermore let us recall that for certain types of domains Ω we can extend a holomophic function f defined on a subvariety of Ω in such a way that the extension preserves regularity properties of the original function. Henkin [Hen72] proved that if Ω is strictly pseudoconvex then a bounded function f defined of a subvariety of Ω can be extended to a function in $H^\infty(\Omega)$. Later on Henkin and Polyakov [HP84] proved the analogous theorem for subvarieties of a polydisc. Likewise Amar [Ama84] proved that if Ω has C^∞-smooth boundary a function which belong to A^∞ on a subvariety of Ω can be extended ta a function in $A^\infty(\Omega)$.

Theorem 3.8. *Let Ω be a ω-convexifiable domain and $\widehat{\Omega} \subset \mathbb{C}^m$ an inflation of Ω such that $\widehat{\Omega}$ has at least $C^{1+\epsilon}$-boundary, $0 < \epsilon < 1$. Further assume that there exists an extension operator $L : \mathcal{A}(\Omega) \to \mathcal{A}(\widehat{\Omega})$.*

If $\mathcal{A}$ is either $A^k(\Omega), 0 \leq k < \infty$ or $H^\infty(\Omega)$, then Ω has the Gleason $\mathcal{A}$-property. For $\mathcal{A} = A^\infty(\Omega)$ Ω has the Gleason $\mathcal{A}$-property, without any assumptions on the boundary smoothness of $\widehat{\Omega}$.

Proof. Let p be a point in Ω and $f \in \mathcal{A}$ be such that $f(p) = 0$.

Using the extension operator L we can extend the function $f \circ \Phi^{-1}(z)$ to a function $\tilde{f} = L(f)$ in $\mathcal{A}(\widehat{\Omega})$ such that $\tilde{f}(P) = 0$, where $P = \Phi(p)$.

By theorem 3.9, 3.10 or 3.12 below $\widehat{\Omega}$ has the Gleason $\mathcal{A}$-property and so

$$\tilde{f} = \sum_i g_i(z)(z_i - P_i).$$

This means that

$$f = \sum_i g_i(\Phi(z))(\phi_i(z) - P_i)$$

where $g_i(\Phi(z)) \in \mathcal{A}(\Omega)$ and the $(\phi_i(z) - P_i)$ are holomorphic on a neighbourhood of Ω.

Since Ω is ω-convexifiable we know from theorem 3.2 that there is a smooth strictly pseudoconvex set $\Omega_2 \supset\supset \Omega$ such that $(\phi_i(z) - P_i) \in H(\Omega_2)$. Now by theorem 3.11 Ω_2 has the Gleason $H(\Omega_2)$-property and so

$$(\phi_i(z) - P_i) = \sum_j h_{ij}(z)(z_j - p_j),$$

where $h_{ij} \in H(\Omega_2)$.

Using some simple algebra we get

$$f = \sum_i g_i(\Phi(z))(\sum_j h_{ij}(z)(z_j - p_j)) = \sum_i k_i(z)(z_i - p_i),$$

with $k_i \in \mathcal{A}$.

□

I would here like to conjecture that the theorem holds for all convexifiable domains. In particular that any convex domain has the Gleason $\mathcal{A}$-property for the mentioned algebras. That Ω has the Gleason A^ω-property is immediate from Theorem 3.2 and 3.11.

We say that a domain Ω has $C^{1+\epsilon}$-boundary if there is a defining function $\sigma(z)$ for Ω such that $\sigma \in C^1(\Omega)$ and each first order derivative of σ satisfies a Hölder-ϵ condition.

Theorem 3.9 (Backlund-Fällström)**.** [BF95] *If Ω is a convex domain in $\mathbb{C}^n$ with $C^{1+\epsilon}$ boundary then Ω has the Gleason $\mathcal{A}$-property for both $A(\Omega)$ and $H^\infty(\Omega)$.*

Theorem 3.10 (Lemmers)**.** [Lem02] *If Ω is a convex domain in $\mathbb{C}^n$ with $C^{1+\epsilon}$ boundary then Ω has the Gleason $\mathcal{A}$-property for $A^k(\Omega), 0 \leq k \leq \infty$.*

Theorem 3.11 (Oka-Hefer)**.** [Oka41] *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$, p a point in Ω and $f \in H(\Omega)$ such that $f(p) = 0$, then there exists $g_1, \ldots, g_n \in H(\Omega)$ such that*

$$f(z) = \sum g_i(z)(z - p).$$

Theorem 3.12. *If Ω is a convex domain in $\mathbb{C}^n$ then Ω has the Gleason $\mathcal{A}$-property for $A^\infty(\Omega)$.*

Proof. Our proof will be based on the method of Leibenzon [Hen71]. Let p be a point in Ω, without loss of generality we can assume that $p = 0$.

Following Leibenzon we note that

$$f(z) = \int_0^1 \frac{d}{d\lambda} f(\lambda z) d\lambda = \int_0^1 \sum_{i=1}^n z_i \left(D_i f\right)(\lambda z) d\lambda =$$

$$= \sum_{i=1}^n z_i \int_0^1 \left(D_i f\right)(\lambda z) d\lambda = \sum_{i=1}^n z_i T_i(f, z).$$

Thus we have a solved the Gleason problem if we can show that

$$T_i(f, z) = \int_0^1 \left(D_i f\right)(\lambda z) d\lambda$$

is a function in A^∞ whenever $f \in A^\infty$.

Now we have that

$$\left| \frac{\partial}{\partial z_j} T_i(f, z) \right| = \left| \frac{\partial}{\partial z_j} \int_0^1 \left(D_i f\right)(\lambda z) d\lambda \right| \leq \sup_{z \in \Omega} \left| \frac{\partial^2}{\partial z_j \partial z_i} f(z) \right|.$$

Thus if $\frac{\partial^2}{\partial z_j \partial z_i} f$ is bounded for all i, j then each first derivative of $T_i(f, z)$ will be bounded and so $T_i(f, z)$ will be continuous. This implies that if $f \in A^k$ then $T_i(f, z) \in A^{k-2}$. However for $f \in A^\infty$ this means that $T_i(f, z) \in A^\infty$.

□

In this section we have seen that some of the good properties of convex domains give rise to corresponding good properties for convexifiable domains. There are however some things that do not work out quite as nicely. For example the Monge-Ampàre operator of a function on the inflation might be zero but its pullback to the original domain is most likely not. We can however expect properties of a more algebraic character to transform well.

4. Convexifiability of complex domains

In the introduction we mentioned the theorem by Fornaess which started this investigation and also gave an example of a convexifiable domain not covered by this theorem. In this section we will give more examples of classes of convexifiable domains and ways to construct new domains from other convexifiable domains.

Our first two theorems mimics two basic theorems about pseudoconvex domains and gives us two operations under which the property of convexifiability is closed.

Theorem 4.1. *The cartesian product $\Omega_3 = \Omega_1 \times \Omega_2$ of a k-convexifiable domain Ω_1 and a j-convexifiable domain Ω_2 is an m-convexifiable domain, $m = \min\{\, j, k \,\}$, $\infty \prec \omega$.*

Proof. Let $\widehat{\Omega}_1$ be an inflation of Ω_1 and ϕ_1 the mapping connecting them, $\widehat{\Omega}_2$ and ϕ_2 analogously for Ω_2.

Now let $\widehat{\Omega}_3 = \widehat{\Omega}_1 \times \widehat{\Omega}_2$ and $\phi_3 = (\phi_1, \phi_2)$. The domain $\widehat{\Omega}_3$ is convex, ϕ_3 is proper, since if (z, w) is mapped into the boundary of Ω_3 then either z or w belongs to the boundary of Ω_1 or Ω_2 respectively, and has the claimed regularity. We thus have an inflation of Ω_3 $\square$

Theorem 4.2. *The intersection $\Omega_3 = \Omega_1 \cap \Omega_2$ of a k-convexifiable domain Ω_1 and a j-convexifiable domain Ω_2 is an m-convexifiable domain, $m = \min\{j, k\}$, $\infty \prec \omega$.*

Proof. Choose $\widehat{\Omega}_3$ and ϕ_3 as in the proof of Theorem 4.1. ϕ_3 is proper since if (z, w) belongs to the boundary of Ω_3 then at least one of z and w must belong to the boundary of Ω_1 and Ω_2 respectively, and thus to boundary of $\Omega_1 \cap \Omega_2$. $\square$

The previous theorem should also hold in the form that if Ω_1 and Ω_2 are both strictly convexifiable then Ω_3 is strictly convexifiable as well. I do not have a presentable proof of this so I leave this a conjecture.

In the same spirit we also have the following proposition.

Proposition 4.3. *Let Ω be a (strictly) k-convexifiable domain and V a holomorphic subvariety of Ω. Then V is also (strictly) k-convexifiable.*

Proof. Let $\widehat{\Omega}$ be an inflation of Ω and Φ the mapping between them.

Now just use the restriction of Φ to V and the same set $\widehat{\Omega}$ as an inflation of V. $\square$

Next we recall Ex 2.1 in which we saw how to convexify an annulus. A generalisation of that example shows that a large number of domains in $\mathbb{C}$ are also convexifiable.

Theorem 4.4. *Let $\Omega \subset \mathbb{C}$ be a domain of genus $k < \infty$ with C^j-boundary such that no component of its complement is a point. Then Ω is j-convexifiable and the polydisc in $\mathbb{C}^{k+1}$ can be chosen as an inflation of Ω.*

Proof. Let $C_0, C_1, \ldots, C_{k-1}$ be the contours bounding Ω and choose the order so that C_1 is the outer boundary.

Let ϕ_0 be a biholomorphism of Ω that takes C_0 to a fixed circle C and $\phi_i,\ i > 0$ a biholomorphism of Ω that takes C_0 to C and C_i to a circle concentric to C and of smaller radius. Such biholomorphisms exist and belong to $C^j(\Omega)$, see for example [Ahl79]

Now let $\Phi = (\phi_0, \frac{1}{\phi_1}, \frac{1}{\phi_2}, \ldots, \frac{1}{\phi_k})$. Since the modulus of ϕ_i is constant on C_i Φ maps Ω into the $(k+1)$-dimensional polydisc which becomes an inflation of Ω. $\square$

By Theorem 3.6 this means that $H(\Omega)$ has a set of $k+1$ generators and using your favourite proof of Runge's theorem this can also be seen to be the minimum number needed.

In the previous theorem we excluded domains with points removed from them and as small sidetrack we will give a sufficient condition for a domain not to be convexifiable. From this proposition we will see that the conditions of the previous theorem are in fact not only sufficient but also necessary.

Proposition 4.5. *If the complement of a pseudoconvex domain $\Omega \subset \mathbb{C}^n$ contains a holomorphic subvariety V such that the intersection of V with the interior of $\overline{\Omega}$ is non-empty then Ω can not be convexified.*

Proof. Let $S = \overline{\Omega} \setminus \partial\overline{\Omega}$. If it was possible to convexify Ω then for each point $p \in \partial\Omega$ there would exist a non-constant bounded holomorphic function such that $\sup_{z\in\Omega} |f(z)| \leq |f(p)|$. Let f be such a weak peak function for a point $p \in S$. By theorem K3 of [Gun90b] this function can be extended over V to a function holomorphic on S. Now this function would have a local maximum at p and must be constant.

We have a contradiction. $\square$

For further conditions under which a domain can not be convexifiable see [Noe91] where, as an example, it is shown that the "worm"-domain of Diederich and Fornaess can not be convexified by a smooth enough mapping.

Generalising further from example 2.1 we have the following.

Theorem 4.6. *All analytic polyhedra* $\Omega = \{z : \ |f_j(z)| < 1, \forall f_j, j = 1, \dots, k, \ \ f_j \in H(W), \Omega \subset\subset W\}$ *in* $\mathbb{C}^n$ *can be* ω*-convexified into the* $k+n$*-dimensional polydisc.*

Proof. Choose c such that $|cz| < 1$, $\forall z \in \bar{\Omega}$ and let $\Phi = (cz, f_1, f_1, \dots, f_k)$. At any point on the boundary of Ω there will be one or more function having modulus 1 and the rest will have a modulus smaller than one. Thus every point in $\partial\Omega$ will be mapped into the boundary of the $(n+k)$-dimensional polydisc and the interior points into its interior. The mapping Φ will have the required smoothness since the functions f_j are all holomorphic on some neighbourhood of Ω. □

Now, as the finale among the theorems of this section, we cite the original theorem by Fornaess concerning strictly pseudoconvex sets.

Theorem 4.7 (Fornaess)**.** [For76] *Let* $G \subset\subset X \subset\subset \mathbb{C}^n$ *be a bounded, strictly pseudoconvex domain with* C^k *boundary,* $k \geq 2$*, and* X *a bounded pseudoconvex domain. Then there exist a holomorphic map* $\psi : X \to \mathbb{C}^m$ *for some* $m \geq n$ *and a convex, strictly pseudoconvex, bounded domain* C *with* C^k *boundary in* $\mathbb{C}^m$ *such that*

(1) ψ *is biholomorphic onto a closed subvariety of* $\mathbb{C}^m$*,*

(2) $\psi(G) \subset C$ *and* $\psi(X \setminus \bar{G}) \subset \mathbb{C}^m \setminus \bar{C}$ *and*

(3) $\psi(X)$ *intersects* ∂C *transversally*

In our terminology this would mean that G is strictly ω-convexifiable.

We note the stark contrast between the situation in Theorem 4.1, 4.4 and 4.6 which tell us that all analytic polyhedra and all poly-domains can be convexified into the polydisc, a Levi-flat domain and Theorem 4.7 which maps strictly pseudoconvex sets into domains which are both convex and strictly pseudoconvex. In fact none of the above theorems gives us any large class of domains with inflations which are weakly pseudoconvex but not Levi-flat.

It would of course be interesting to find out more about which classes of domains are convexifiable. Corollary 3.5 tells us that smooth strictly convexifiable domain has a dense set of strictly pseudoconvex boundary points. That almost everywhere strictly pseudoconvex boundary is not a sufficient condition for convexification can be seen from the domain $\{(z,w) : |z|^2 + (|w|+1)^2 < 1\}$, a slight variation of the Hartogs triangle, the boundary of which is strictly pseudoconvex except at the origin. This domain is however not smooth and it would be interesting to find out whether smoothness combined with some condition weaker than strict pseudoconvexity implies convexifiability. For domains which are not C^2-smooth one possibility would be that the Levi-form is defined and uniformly bounded from below at a dense set of strictly pseudoconvex boundary points, as is the case with the intersection of two strictly pseudoconvex domains.

This question gains some interest from N Sibony's construction of a smooth domain, the boundary of which is strictly pseudoconvex except at one point, on which the corona problem for H^∞ fails, see [Sib87]. The existence of domains such as this together with extension theorems such as those mentioned in connction with Theorem 3.8 raises the interesting possibility that there might exist convex domains on which the corona problem for H^∞ fails.

Acknowledgments

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