

Closure Properties and Negatively Associated Measures violating the van den Berg-Kesten Inequality

Klas Markström

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Abstract

We first give an example of a negatively associated measure which does not satisfy the van den Berg-Kesten inequality. Next we show that the class of measures satisfying the van den Berg-Kesten inequality is not closed under either of conditioning, introduction of external fields or convex combinations. Finally we show that this class also includes measure which have rank sequence which is not logconcave.

1 Introduction

In 2009 Petter Brändén gave lecture series on the results of [BBL09] at the Newton Institute. After the one of the lectures the following problem came up during a discussion between the current author and Petter Brändén, Jeff Kahn and Rob van den Berg: *Construct an example of a measure which is negatively associated but does not satisfy the van den Berg Kesten-Inequality*. Note that all measures satisfying the BK-inequality are negatively associated [DR98]. In fact this had been stated as an open problem already in 1998 by Dubhashi and Ranjan [DR98]. At the end of the workshop the author had found one such example and the first aim of this paper is to give the construction. In fact we will give a simpler example than the original one.

In [BBL09] Borcea, Brändén and Liggett considered the class of Strongly Rayleigh measures, the Rayleigh measures were introduced by [Wag08] in the context of matroid theory, for which they could show a number of correlation results, extending negative association, and also that the class was closed under several operations earlier discussed in [Pem00] in connection with general negatively associated measures. Two of these operations are conditioning on a variable and introducing an external field. The second aim of this paper is to construct a measure which shows that the class of measurers satisfying the BK-inequality is not closed under conditioning or external fields. This means that even though the class of measure satisfying the BK-inequality have strictly stronger negative correlation properties than a general negatively associated

measure they are not as robust as the strongly Rayleigh measures. Whether a strongly Rayleigh measure in turn must satisfy the BK-inequality or not is currently unknown.

We begin with some definitions. Let B_n denote the boolean lattices of subsets of $1, \dots, n$, equivalently this can be interpreted as the set of binary strings of length n in the standard way. Let $X = (X_1, \dots, X_n)$ denote a random binary vector from B_n distributed according to a measure μ on B_n .

Definition 1.1. Two functions f and g depend on disjoint set of variables if there exist a set of indices A such that f only depends on the X_i with $i \in A$ and g depends only on X_j with $c \in (\{1, \dots, n\} \setminus A)$

A probability measure μ on B_n is negatively associated if every pair of increasing functions $f : B_n \rightarrow \mathbb{R}^+$, $g : B_n \rightarrow \mathbb{R}^+$ such that f and g depend on disjoint sets of variables satisfies

$$\int fg d\mu \leq \int f d\mu \int g d\mu \quad (1)$$

Since any increasing function can be written as a linear combination of indicator functions of upsets we see that it suffices for condition 1 to hold for all such pairs of indicator functions.

Recall that an event is increasing if its indicator function is increasing. Next we need to define the concept that two events A and B occur disjointly, denoted $A \square B$. This is most easily done by interpreting the the boolean lattice B_n in terms of sets. Then $A \square B = \{C \in B_n | \exists A' \subset C, \exists B' \subset C, A' \cap B' = \emptyset, \text{ such that for any } D \in B_n : A' \subset D \Rightarrow D \in A, B' \subset D \Rightarrow D \in B\}$. This can be interpreted as saying that A happens in A' and B in the set B' which is disjoint from A' .

Definition 1.2. A probability measure μ on B_n satisfies the van den Berg-Kesten, abbreviated BK, inequality if any pair of increasing events A and B satisfies

$$\mu(A \square B) \leq \mu(A)\mu(B) \quad (2)$$

This inequality was first studied in [vdBK85] where it was proven that the product measure on B_n satisfies the BK-inequality. This result has proven to be one of the most important tools in the study of both percolation and random graphs. Those authors also conjectured that for product measures the inequality holds for all pairs of events, not just increasing ones. This conjecture was proven only recently [Rei00]. However there are still numerous unsolved basic questions regarding the class of measures which satisfies the BK-inequality. One of the most fundamental such question is the conjecture by Dubhashi and Ranjan that the class of BK-measures is closed under direct products [DR98].

2 Negative association does not imply the BK-inequality

We now give two examples of measures which are negatively associated but do not satisfy the BK-inequality. We will describe our measures in terms of the probabilities for a binary string $x_1x_2x_3$, which can be seen as the characteristic function of a set in B_3 in the standard way.

Example 2.1.

$$\begin{aligned} p(111) &= 0 \\ p(110) &= \frac{294}{1000}, \quad p(101) = p(011) = \frac{10}{1000} \\ p(001) &= \frac{6}{1000}, \quad p(010) = p(100) = \frac{240}{1000} \\ p(000) &= \frac{200}{1000} \end{aligned}$$

The first example is in fact sufficiently robust that some mass can be moved to make the measure strictly positive.

Example 2.2.

$$\begin{aligned} p_2(111) &= \frac{2}{1000} \\ p_2(110) &= \frac{294}{1000}, \quad p_2(101) = p_2(011) = \frac{10}{1000} \\ p_2(001) &= \frac{6}{1000}, \quad p_2(010) = p_2(100) = \frac{240}{1000} \\ p_2(000) &= \frac{198}{1000} \end{aligned}$$

For these examples we have have the following result.

Theorem 2.3. *The measures in Examples 2.1 and 2.2 are negatively associated but do not satisfy the BK-inequality.*

Verifying that these measures satisfy Definition 1.1 is simplified by the fact that the measures are invariant under exchange of x_1 and x_2 , but the calculation is still so long that we will here only verify it for the first example, where $p(111) = 0$ simplifies the calculations further. For both examples we have verified the calculations by computer algebra as well.

Proof. For $n = 3$ there are 9 pairs of increasing events for which we need to check the condition of Definition 1.1. Due to the symmetries of the measures the number of distinct pairs is smaller.

1. A pair of events of the form $x_i = 1$:

$$\begin{aligned}
p(x_1 = 1) &= p(x_2 = 1) = \frac{1}{1000}(240 + 10 + 294) = \frac{544}{1000} \\
p(x_3 = 1) &= \frac{1}{1000}(6 + 10 + 10) = \frac{26}{1000} \\
p(x_1 = 1 \wedge x_3 = 1) &= p(x_2 = 1 \wedge x_3 = 1) = \frac{1}{1000}(10 + 0) = \frac{10}{1000} \leq \\
&\leq p(x_1 = 1)p(x_3 = 1) = \frac{221}{15625} \approx 0.0141 \\
p(x_1 = 1 \wedge x_2 = 1) &= \frac{1}{1000}(294 + 0) = \frac{294}{1000} \leq p(x_1 = 1)p(x_2 = 1) = \\
&= \frac{4624}{15625} \approx 0.2959
\end{aligned}$$

2. A single variable and event of the form $(x_i = 1 \wedge x_j = 1)$: For i, j, k all distinct:

$$p((x_i = 1 \wedge x_j = 1) \wedge (x_k = 1)) = p(x_1 = 1 \wedge x_2 = 1 \wedge x_3 = 1) = 0 \quad (3)$$

3. The events $(x_1 = 1 \vee x_2 = 1)$ and $x_3 = 1$:

$$\begin{aligned}
p(x_1 = 1 \vee x_2 = 1) &= \frac{1}{1000}(240 + 240 + 10 + 10 + 294) = \frac{794}{1000} \\
p((x_1 = 1 \vee x_2 = 1) \wedge (x_3 = 1)) &= p(x_1 = 1 \wedge x_3 = 1) + p(x_2 = 1 \wedge x_3 = 1) + 0 = \\
&= \frac{1}{1000}(10 + 10 + 0) = \frac{20}{1000} \leq p((x_1 = 1 \vee x_2 = 1))p((x_3 = 1)) = \\
&\qquad\qquad\qquad \frac{5161}{250000} \approx 0.0206
\end{aligned}$$

4. For distinct i, j , the events $(x_i = 1 \vee x_3 = 1)$ and $x_j = 1$:

$$\begin{aligned}
p(x_1 = 1 \vee x_3 = 1) &= p(x_2 = 1 \vee x_3 = 1) = \frac{1}{1000}(240 + 6 + 10 + 10 + 294 + 0) = \\
&= \frac{560}{1000} \\
p((x_2 = 1 \vee x_3 = 1) \wedge (x_1 = 1)) &= p((x_1 = 1 \vee x_3 = 1) \wedge (x_2 = 1)) = \\
&= p(x_1 = 1 \wedge x_2 = 1) + p(x_2 = 1 \wedge x_3 = 1) + 0 = \frac{1}{1000}(294 + 10) = \\
&= \frac{304}{1000} \leq p((x_1 = 1 \vee x_3 = 1))p((x_2 = 1)) = \frac{952}{3125} \approx 0.3046
\end{aligned}$$

This concludes the proof that the measure is negatively associated.

In order to prove that the measure does not satisfy a BK-inequality it suffices to give an explicit pair of events for which the inequality fails. For this measure

the events $(x_1 = 1 \vee x_3 = 1)$ and $(x_2 = 1 \vee x_3 = 1)$ gives

$$\begin{aligned} p((x_1 = 1 \vee x_3 = 1) \square (x_2 = 1 \vee x_3 = 1)) &= \frac{1}{1000}(10 + 10 + 294 + 0) = \frac{314}{1000} \\ p((x_1 = 1 \vee x_3 = 1))p((x_2 = 1 \vee x_3 = 1)) &= \frac{560^2}{1000^2} = \frac{196}{625} = 0.3136 \quad (4) \end{aligned}$$

□

3 The BK-inequality is not preserved under conditioning or external fields

We first need two definitions which are extensions of properties considered for negatively associated measures in [Pem00] to measures satisfying the BK-inequality.

Definition 3.1. A measure μ on B_n is *conditionally BK* if all measures, on B_{n-1} , of the form $\mu(A|x_i = j)$, for $i = 1, \dots, n$ and $j = 0, 1$ satisfy the BK-inequality.

Definition 3.2. Let $W : \{1, \dots, n\} \rightarrow \mathbb{R}^+$ be a non-negative weight function and let μ' be the reweighted measure obtained from μ as

$$\mu'(X = (y_1, y_2, \dots, y_n)) = C\mu(X = (y_1, \dots, y_n)) \prod_i W(i)^{y_i},$$

where C is a constant normalising the total measure to 1.

Note that taking $W(i) = 1$ for all values of i except j and then letting $W(j)$ go to 0 or ∞ corresponds to conditioning on x_j being 1 and 0 respectively. We call the latter the extremal external fields.

Our third example is again a modification of Example 2.1. In this case we have moved some mass in order to make the measure satisfy the BK-inequality.

Example 3.3.

$$\begin{aligned} q(111) &= 0 \\ q(110) &= \frac{294}{1000}, \quad q(101) = q(011) = \frac{10}{1000} \\ q(001) &= \frac{7}{1000}, \quad q(010) = q(100) = \frac{240}{1000} \\ q(000) &= \frac{199}{1000} \end{aligned}$$

We now have

Theorem 3.4. *The measure in Example 3.3 satisfies the BK-inequality but is not conditionally BK and does not satisfy the BK-inequality for some non-extremal external fields.*

Proof. The proof breaks into three parts:

1. Checking that the measure satisfies the BK-inequality in this case reduces to verifying that the measure is negatively associated and satisfies three inequalities of the form used in equations 4. Checking that the measure is negatively associated is easy and only a small modification of the calculations done in the proof of Theorem 2.3 so we will leave them out and instead verify the final three inequalities.

Again the calculations are simplified by the symmetries of the measure.

$$\begin{aligned} q((x_1 = 1 \vee x_2 = 1) \square (x_1 = 1 \vee x_3 = 1)) &= q((x_1 = 1 \vee x_2 = 1) \square (x_2 = 1 \vee x_3 = 1)) = \\ &= \frac{1}{1000}(10 + 10 + 294 + 0) = \frac{314}{1000} \\ q((x_1 = 1 \vee x_2 = 1))q((x_1 = 1 \vee x_3 = 1)) &= q((x_1 = 1 \vee x_2 = 1))q((x_2 = 1 \vee x_3 = 1)) = \\ &= \frac{240 + 240 + 10 + 10 + 294}{1000} \frac{240 + 7 + 10 + 10 + 294}{1000} = \frac{222717}{500000} \approx 0.44. \end{aligned}$$

The two inequalities just verified do in fact hold for Example 2.1 as well. The more critical inequality is the one which fails for Example 2.1

$$\begin{aligned} q((x_1 = 1 \vee x_3 = 1) \square (x_2 = 1 \vee x_3 = 1)) &= \frac{1}{1000}(10 + 10 + 294 + 0) = \frac{314}{1000} \\ q((x_1 = 1 \vee x_3 = 1))q((x_2 = 1 \vee x_3 = 1)) &= \frac{(240 + 7 + 10 + 10 + 294)^2}{1000^2} = \\ &= \frac{314721}{1000000} \approx 0.3147 \end{aligned}$$

2. To see that this measure is not conditionally BK we look at the case where we condition on the event $x_3 = 0$. The conditioned measure is then

$$\begin{aligned} q_3(111) &= 0 \\ q_3(110) &= \frac{294}{973}, \quad q_3(101) = q_3(011) = 0 \\ q_3(001) &= 0, \quad q_3(010) = q_3(100) = \frac{240}{973} \\ q_3(000) &= \frac{199}{973} \end{aligned}$$

Here we find that

$$q_3(x_1 = 1)q_3(x_2 = 1) = \frac{(240 + 294)^2}{973^2} \approx 0.3012$$

and

$$q_3(x_1 = 1 \wedge x_2 = 1) = \frac{294}{973} \approx 0.3021$$

Which means that the conditioned measure violates the BK-inequality for the pair of events $x_1 = 1$ and $x_2 = 1$.

3. Since the inequality we found in the previous case was a strict inequality we can use that the introduction of of an external field is a continuous transformation of the measure, and the relation to conditioning mentioned earlier, to deduce that there must exist non-extremal fields of the form $W(x_1, x_2, x_3) = s^{x_3}$ for which the measure does not satisfy the BK-inequality.

More explicitly, if we consider the weight $W(x_1, x_2, x_3) = s^{x_3}$ and use a computer algebra package to reduce the set of inequalities in the definition of the BK-inequality we find that the reweighted measure satisfies the BK-inequality if and only if $s \geq \frac{1}{189}(-719 + \sqrt{688195}) \approx 0.585$

□

4 Convex Combinations

Recall that a measure μ_1 stochastically dominates a measure μ_2 if $\mu_1(A) \geq \mu_2(A)$ for all increasing events A . We denote this by $\mu_1 \succ \mu_2$. It is well known, see e.g. chapter 2 of [Lig85], that if μ_1 and μ_2 satisfy the FKG-inequality and $\mu_1 \succ \mu_2$ then every convex combination $t\mu_1 + (1-t)\mu_2$, $0 \leq t \leq 1$ also satisfies the FKG-inequality. Without the stochastic domination condition this does not hold.

Our final example shows that convex combinations do not preserve the BK-inequality, even for product measures.

Example 4.1. We will consider two product measures on B_2 . Let μ_1 be the product measure with $\mu_1(x_1 = 1) = \frac{8}{10}$ and μ_2 be the product measure with $\mu_2(x_1 = 1) = \frac{1}{10}$.

Let us now consider the event $A = (x_1 = 1 \vee x_2 = 1)$.

$$(t\mu_1 + (1-t)\mu_2)(A) = \frac{24t}{25} + \frac{19(1-t)}{100}$$

$$(t\mu_1 + (1-t)\mu_2)(A \square A) = (t\mu_1 + (1-t)\mu_2)(x_1 = 1 \wedge x_2 = 1) = \frac{64t}{100} + \frac{1-t}{100}$$

We may now solve the inequality

$$\frac{64t}{100} + \frac{1-t}{100} \leq \left(\frac{24t}{25} + \frac{19(1-t)}{100} \right)^2$$

for t . Some basic algebra gives a quadratic equation in t and we find that the inequality is satisfied if and only if

$$t \leq \frac{1}{847}(241 \pm 10\sqrt{265}) \leq t$$

meaning that t must be less than approximately 0.08 or greater than approximately 0.48.

Hence the measure $t\mu_1 + (1-t)\mu_2$ does not satisfy the BK-inequality for an interval of values of t .

5 Log-Concavity for the rank sequence

In [Pem00] a number of conjectures regarding the properties of negatively associated measures were given, in particular the conjecture that the rank sequence of such a measure should be ultra-logconcave received a lot of attention due to its connection with Mason's conjecture [Wag08]. Recall that the rank sequence is the numbers $r_i = \sum \mu(A)$, where the sum runs over all sets A with cardinality i . In [Mar07] the author constructed a minimum counterexample to this conjecture, and in [BBL09] several families of such examples were given.

Our final example is a measure which satisfies the BK-inequality but has a rank sequence which is not logconcave, a property strictly weaker than ultra-logconcavity. Verifying the claimed properties is done exactly as for our earlier measures and we leave the calculations out.

Example 5.1.

$$\begin{aligned}\mu(000) &= \mu(\{111\}) = \frac{1}{100} \\ \mu(100) &= \mu(010) = \frac{30}{100} & \mu(001) &= \frac{32}{100} \\ \mu(110) &= \mu(101) = \mu(011) = \frac{2}{100}\end{aligned}$$

1/100 92/100 6/100 1/100

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